THE TODA LATTICE

Background

The Toda lattice is named after Morikazu Toda, who discovered in the 1960's that the differential equations for a lattice with internal force

$$T(y) = \alpha(e^{\beta y} - 1)$$

(with α, β constant) admit solutions which can be written in terms of elliptic functions. This extends the fact that the standard lattice with linear internal force $T(y) = ky$ can be solved using trigonometric functions.

Since nonlinear o.d.e. are usually much more complicated than linear o.d.e., the very fact that it admits explicit solutions at all means that the Toda lattice is already a remarkable example. (The Fermi-Pasta-Ulam lattice, in contrast, has no analogous explicit solutions.) A few years later, a possible explanation of this property appeared, when it was discovered that the Toda lattice is an example of a "completely integrable Hamiltonian system". This, in turn, led to an explanation of the unexpectedly simple behaviour of the Fermi-Pasta-Ulam lattice (see the ATO for the Fermi-Pasta-Ulam lattice): on the one hand, for a completely integrable Hamiltonian system, (almost) periodic behaviour can be predicted, and on the other hand, the Fermi-Pasta-Ulam lattice can be regarded as an approximation to the Toda lattice when the vibrations are small. We shall say more about completely integrable Hamiltonian systems below.

How to view the demonstration

It can be seen that the Toda lattice behaves in a very similar way to the Fermi-Pasta-Ulam lattice. Instead of thermalizing, the lattice motion appears to be periodic.

Further aspects

Mathematically, however, the Toda lattice is much easier to analyse than the Fermi-Pasta-Ulam lattice, because it is a completely integrable Hamiltonian system. This means, practically speaking, that it has the maximum possible number of conserved quantities. For a system of n ordinary differential equations of second order, this maximum number is n. For example, for the equation $y'' = -ky$, the total energy $\frac{1}{2}{y'}^2 + \frac{1}{2}ky^2$ (kinetic energy plus potential energy) is a conserved quantity, i.e. it is constant when y is a solution of the differential equation, and there are no others. The standard linear lattice is of this type; each of the normal mode energies is a conserved quantity, because each represents the total energy of an "uncoupled" (independent) oscillator.

The fact that the Toda lattice has the maximum possible number of conserved quantities is not obvious, and certainly not on physical grounds. A mathematical explanation comes from the fact that the equations of motion may be written in the form $L' = [L, M]$, where L and M are matrix functions. This type of equation is called a Lax equation (after Peter Lax), and the Lax equation for the Toda lattice was discovered by Hermann Flaschka in the 1970's. For any Lax equation, the eigenvalues of the matrix function L turn out to be conserved quantities, and this gives the required number of conserved quantities of the Toda lattice (though their physical meaning remains unclear).

The conserved quantities greatly constrain the motion of the system. In fact, the Arnold-Liouville Theorem says that the motion of an n-dimensional completely integrable Hamiltonian system must, if it is bounded (as in our case), be equivalent to linear motion on an n-dimensional torus. When $n = 1$ this means that the motion must be periodic. When $n = 2$ (and similarly if $n > 2$) the motion must be expressable as $(e^{2\pi\sqrt{-1}\,at}, e^{2\pi\sqrt{-1}\,at})$ for some real numbers a, b; if a/b is rational the motion is periodic, otherwise it is almost periodic in the sense that it winds densely around the torus, returning arbitrarily closely to its initial value.

Although the Fermi-Pasta-Ulam lattice and the Toda lattice are approximately the same when the vibrations are small, it can be shown that the Fermi-Pasta-Ulam lattice is not a completely integrable Hamiltonian system. Nevertheless, the almost periodic motion of the Toda lattice is inherited by the Fermi-Pasta-Ulam lattice, at least for small vibrations, and this explains the absence of thermalization in both cases.

Completely integrable Hamiltonian systems are important, but quite special. It is an important area of current research to identify and study more general types of "integrable systems".

References

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