A Crash Course of Floer Homology for Lagrangian Intersections

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1 Introduction

There are several kinds of Floer homology:

- Floer homology for Lagrangian intersections [1],
- Floer homology for fixed points of symplectic diffeomorphisms [2],
- Floer homology in Yang-Mills theory [3].

Beyond them there exists Morse theory or Morse homology for infinite dimentional manifolds. In this manuscript we will explain the theory with respect to the first case.

2 Morse homology

Let M be a closed manifold of finite dimension and $f : M \to \mathbf{R}$ a smooth function. We call a point p *critical* if and only if $df_p = 0$. Then we can obtain a symmetric bilinear form $Hf_p : T_pM \times T_pM \to \mathbf{R}$ called the *Hessian*. The definition of Hf_p is

$$Hf_p(u,v) := \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(p) a_i b_j,$$

where $u = \sum_{i=1}^{n} a_i (\frac{\partial}{\partial x_i})_p$ and $v = \sum_{j=1}^{n} b_j (\frac{\partial}{\partial x_j})_p$. If Hf_p is non-degenerate, then we call the critical point *non-degenerate*, and we call the number of the negative eigenvalues of

$$\left(\frac{\partial^2 f}{\partial x_i \partial x_j}(p)\right)_{1\le i,j\le n}$$

the *Morse index* of f at p. We shall use $\mu(p)$ to denote the index. If all the critical points of f are non-degenerate, then we call f a *Morse function.* In fact, there are many Morse functions on M.

Let g be a Riemannian metric on M. For $f : M \to \mathbf{R}$ there exists a unique vector field grad$f$ on M such that

$$g(v, \mathrm{grad} f) = v(f).$$

We call grad$f$ the *gradient vector field* of f. For critical points p and q we consider the set

$$\mathcal{M}(p,q) := \{x : \mathbf{R} \to M | \dot{x} = -\mathrm{grad} f, \lim_{\tau\to-\infty} x(\tau) = p, \lim_{\tau\to\infty} x(\tau) = q\}.$$

Note that $\mathbf{R}$ acts on $\mathcal{M}(p,q)$ by $(a\cdot x)(\tau) := x(\tau + a)$ for $a \in \mathbf{R}$. We denote the quotient by $\hat{\mathcal{M}}(p,q)$.

Theorem 2.1 *For generic Riemannian metrics $\hat{\mathcal{M}}(p,q)$ is a smooth manifold of dimension $\mu(p) - \mu(q) - 1$.*

Moreover the following compactification theorem holds.

Theorem 2.2 (1) *If $\mu(p) - \mu(q) - 1 = 0$, then $\hat{\mathcal{M}}(p,q)$ is compact.*
(2) *If $\mu(p) - \mu(q) - 1 = 1$, then we have a suitable compactification of $\hat{\mathcal{M}}(p,q)$ so that the boundary is*

$$\bigcup_{\mu(r)=\mu(p)-1} \hat{\mathcal{M}}(p,r) \times \hat{\mathcal{M}}(r,q).$$

We can choose a suitable orientation of $\hat{\mathcal{M}}(p,q)$ compatible with the compactification.

Let C_k be the $\mathbf{Z}$-module whose basis elements are the critical points of Morse index k,

$$C_k := \bigoplus_{\mu(p)=k} \mathbf{Z}p.$$

We define a linear map $\partial : C_k \to C_{k-1}$ in terms of the canonical bases by

$$\partial p := \sum_{\mu(q)=k-1} \sharp\hat{\mathcal{M}}(p,q) q,$$

where $\sharp\hat{\mathcal{M}}(p,q)$ counts the elements of $\hat{\mathcal{M}}(p,q)$ with the orientation or sign. Then we can calculate

$$\begin{aligned}\partial\partial x &= \partial \sum_{\mu(y)=\mu(x)-1} \sharp\hat{\mathcal{M}}(x,y)y \\ &= \sum_{\mu(y)=\mu(x)-1}\sum_{\mu(z)=\mu(y)-1} \sharp\hat{\mathcal{M}}(x,y)\sharp\hat{\mathcal{M}}(y,z)z.\end{aligned}$$

From Theorem 2.2 (2) the number

$$\sum_{\mu(y)=\mu(x)-1} \sharp\hat{\mathcal{M}}(x,y)\sharp\hat{\mathcal{M}}(y,z)$$

is zero.

Theorem 2.3 *$\partial^2 = 0$ and the homology is isomorphic to the singular homology of M.*

We call the homology *Morse homology.* Here is an example [4]. (Here we omit the signs. For simplicity the reader may consider the example over $\mathbf{Z}_2$.) Let M be a 2-sphere and f the height function as in Figure 1.

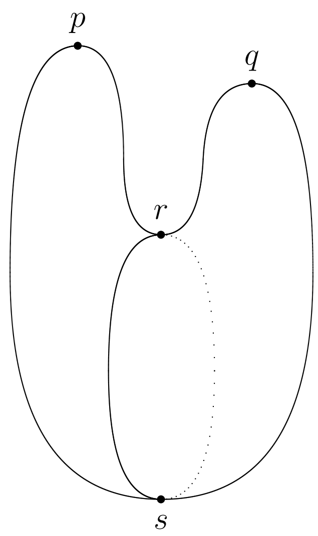

Figure 1

The indices of the critical points are $\mu(p) = \mu(q) = 2$ and $\mu(r) = 1$ and $\mu(s) = 0$. In this case $\hat{\mathcal{M}}(p,r)$ and $\hat{\mathcal{M}}(q,r)$ consist of one point and $\hat{\mathcal{M}}(r,s)$ consists of two points, and we can identify $\hat{\mathcal{M}}(p,s)$ with an open interval and compactify $\hat{\mathcal{M}}(p,s)$ so that the boundary is $\hat{\mathcal{M}}(p,r) \times \hat{\mathcal{M}}(r,s)$ as in Figure 2.

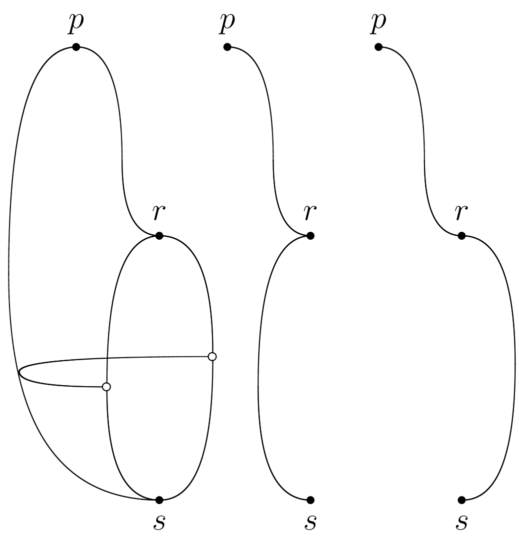

Figure 2

We have

$$\partial p = r,\ \ \partial q = -r,\ \ \partial r = 0,\ \ \partial s = 0,$$

and the homology is

$$\mathbf{Z}[p+q] \oplus \mathbf{Z}[s]$$

which is isomorphic to the singular homology of the sphere.

3 Symplectic preliminaries

Let M be a smooth manifold and ω a 2-form on M. We call ω a *symplectic form* if and only if ω is closed and non-degenerate. From the non-degeneracy we can conclude that the dimension of M is even. We have the following examples of symplectic manifolds:

- $\mathbf{R}^{2n}$ with $\omega := \sum_{i=1}^{n} dx_i \wedge dy_i$,
- Kähler manifolds with Kähler forms,
- Let X be a smooth manifold and $(x_1, \ldots, x_n)$ a local coordinate system. We have a local coordinate system of the cotangent bundle T^*X such that $\sum_{i=1}^{n} y_i dx_i$ corresponds to $(x_1, y_1, \ldots, x_n, y_n)$. Then the 2-form $\omega := \sum_{i=1}^{n} dx_i \wedge dy_i$ is a symplectic form on T^*X.

If a diffeomorphism $f : (M_1, \omega_1) \to (M_2, \omega_2)$ satisfies

$$f^*\omega_2 = \omega_1,$$

then we call f a *symplectomorphism.* For a smooth function $H_t : M \times [0,1] \to \mathbf{R}$ there exists a unique vector field X_t such that

$$\omega(\cdot, X_t) = dH_t.$$

We call X_t a *Hamiltonian vector field.* Let $\{\phi_t\}_{0 \le t \le 1}$ be the isotopy such that

$$\begin{cases} \frac{d}{dt}\phi_t &= X_t \circ \phi_t, \\ \phi_0 &= \mathrm{id}. \end{cases}$$

We call such an isotopy a *Hamiltonian isotopy.* From the definition we can conclude that $\phi_t : M \to M$ is a symplectomorphism.

Let L be an n-dimensional submanifold of M^{2n}. If $\omega|_{TL} = 0$, then we call L a *Lagrangian submanifold.* We have the following examples of Lagrangian submanifolds:

- 1-dimensional submanifolds of Riemann surfaces,
- The zero-section 0_X of T^*X,
- Let $\{\phi_t\}_{0 \le t \le 1}$ be a Hamiltonian isotopy and L a Lagrangian submanifold. Then $\phi_t(L)$ is also a Lagrangian submanifold.

We will use the following theorem to calculate Floer homology.

Theorem 3.1 *If L is a Lagrangian submanifold, then we can choose a neighborhood $N(L)$ of L which is symplectomorphic to a neighborhood $N(0_L)$ of the zero-section of T^*L, where L is identified with the zero-section.*

4 Floer homology for Lagrangian intersections

Let $f : X \to \mathbf{R}$ be a smooth function. We will consider $H := f \circ \pi : T^*X \to \mathbf{R}$, where $\pi : T^*X \to X$ is the projection. For the (t-independent) Hamiltonian isotopy $\{\phi_t\}_{0 \le t \le 1}$ associated to H, $\phi_1(0_X)$ is the graph of df in T^*X, hence we can identify the intersection points of 0_X and $\phi_1(0_X)$ with the critical points of f. Moreover, if f is a Morse function, then they intersect transversally. From the Morse inequalities we can conclude

Theorem 4.1 *Let X be a closed manifold and $f : X \to \mathbf{R}$ a Morse function. Then*

$$\sharp\{0_X \cap \phi_1(0_X)\} \geq \sum_{i=0}^{\dim X} \mathrm{rank} H_i(X).$$

We shall denote by $0'_X$ the deformation of 0_X under an isotopy. In comparison with Theorem 4.1, if 0_X and $0'_X$ intersect transversally, then we have the following estimate, which is the best possible.

$$\sharp\{0_X \cap 0'_X\} \geq \chi(X),$$

where $\chi(X)$ is the Euler number of X. Floer proved the following theorem conjectured by Arnold, which is an extension of Theorem 4.1.

Theorem 4.2 *Let M be a compact symplectic manifold and L a Lagrangian submanifold. We assume $\int_{D^2} u^*\omega = 0$ for $u : D^2 \to X$ such that $u(\partial D^2) \subset L$. If L and $\phi_1(L)$ intersect transversally, then*

$$\sharp\{L \cap \phi_1(L)\} \geq \sum_{i=0}^{\dim L} \mathrm{rank} H_*(L, \mathbf{Z}_2).$$

We shall denote by L' the deformation of L under an isotopy. If L and L' intersect transversally, then we have the following estimate, which is the best possible.

$$\sharp\{L \cap L'\} \geq \chi(L),$$

where $\chi(L)$ is the Euler number of L.

Let M be a closed symplectic manifold and L a Lagrangian submanifold. For a Hamiltonian isotopy $\{\phi_t\}_{0 \leq t \leq 1}$ we define

$$\Omega := \{l : [0,1] \to M | l(0) \in L, l(1) \in \phi_1(L), l \text{ is homotopic to } \phi_t(x_0)\},$$

where $x_0 \in L$ is a fixed point, and we denote the universal covering space of Ω by $\tilde{\Omega}$.

$$\tilde{\Omega} := \{u : [0,1] \times [0,1] \to M | u(\tau, 0) \in L, u(\tau, 1) \in \phi_1(L), u(0,t) = \phi_t(x_0)\}/\text{homotopy}.$$

We introduce a function $F : \tilde{\Omega} \to \mathbf{R}$

$$F(u) := \int_0^1 d\tau \int_0^1 dt\ \omega\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial \tau}\right).$$

Lemma 4.3 *Assume* $\int_{D^2} u^*\omega = 0$ *for* $u : D^2 \to X$ *such that* $u(\partial D^2) \subset L$. *If* $u_0(1,t) = u_1(1,t)$, *then* $F(u_0) = F(u_1)$.

From this lemma we can regard F as a function on Ω, which is our Morse function on an infinite dimensional manifold. The tangent space $T_l\Omega$ at $l \in \Omega$ is

$$T_l\Omega = \{\xi(t) \in l^*TM | \xi(0) \in T_{l(0)}L, \xi(1) \in T_{l(1)}\phi_1(L)\}.$$

(Strictly speaking, we need Sobolev spaces to define infinite dimensional manifolds.) Before calculating the gradient vector field of F, we note the following fact.

Lemma 4.4 *There exist Riemannian metrics* g *and almost complex structures* J *such that*

- $g(u,v) = \omega(u, Jv)$,
- $g(u,v) = g(Ju, Jv)$.

We will use t-dependent g_t and J_t satisfying the above conditions. We define a metric on Ω by

$$(\xi_1, \xi_2) := \int_0^1 g_t(\xi_1(t), \xi_2(t))dt, \ \ \xi_1, \xi_2 \in T_l\Omega.$$

Then we can calculate

$$\begin{aligned} (dF)_l(\xi) &= \int_0^1 \omega\left(\frac{dl}{dt}, \xi\right) dt \\ &= \int_0^1 \omega\left(\frac{dl}{dt}, J_t(-J_t\xi)\right) dt \\ &= \left(\frac{dl}{dt}, -J_t\xi\right) \\ &= \left(J_t\frac{dl}{dt}, \xi\right). \end{aligned}$$

Hence, $\mathrm{grad}F = J_t\frac{dl}{dt}$. (Strictly speaking $J_t\frac{dl}{dt}$ is not an element of $T_l\Omega$ because of the boundary conditions.) Moreover, we can conclude that $(dF)_l = 0$ if and only if $\frac{dl}{dt} = 0$, which implies that l is a constant map to $L \cap \phi_1(L)$. Then we will consider the following sets, for p and $q \in L \cap \phi_1(L)$:

$$\mathcal{M}(p,q) := \left\{ u : \mathbf{R} \to \Omega \,\middle|\, \frac{\partial u}{\partial \tau} = -\mathrm{grad}F, \lim_{\tau\to-\infty} u(\tau, [0,1]) = p, \lim_{\tau\to\infty} u(\tau,[0,1]) = q \right\}.$$

Note that $\mathbf{R}$ acts on $\mathcal{M}(p,q)$ by $(a\cdot u)(\tau,t) := u(\tau+a,t)$ for $a\in\mathbf{R}$. We denote the quotient by $\hat{\mathcal{M}}(p,q)$. We can express $\frac{\partial u}{\partial\tau} = -\mathrm{grad}F$ as

$$\frac{\partial u}{\partial \tau} + J(u(\tau,t))\frac{\partial u}{\partial t} = 0,$$

which is a non-linear elliptic partial differential equation.

Theorem 4.5 *Assume that L and $\phi_1(L)$ intersect transversally. We can assign a number $\mu(p)$ to each $p\in L\cap\phi_1(L)$, and for generic t-dependent almost complex structures the $\hat{\mathcal{M}}(p,q)$ are smooth manifolds of dimension $\mu(p)-\mu(q)-1$.*

The transversality of $L\cap\phi_1(L)$ corresponds to the non-degeneracy of critical points of Morse functions. Moreover the following compactification theorem holds.

Theorem 4.6 *Assume $\int_{D^2} u^*\omega = 0$ for $u : D^2\to X$ such that $u(\partial D^2)\subset L$.*
(1) *If $\mu(p)-\mu(q)-1=0$, then $\hat{\mathcal{M}}(p,q)$ is compact.*
(2) *If $\mu(p)-\mu(q)-1=1$, then we have a suitable compactification of $\hat{\mathcal{M}}(p,q)$ so that the boundary is*

$$\bigcup_{\mu(r)=\mu(p)-1}\hat{\mathcal{M}}(p,r)\times\hat{\mathcal{M}}(r,q).$$

We will construct an analogue of Morse homology. Here we will use coefficients in $\mathbf{Z}_2$. (To construct the chain complex over $\mathbf{Z}$ we need suitable orientations of $\hat{\mathcal{M}}(p,q)$.) Let C_k be the $\mathbf{Z}_2$-vector space over the intersection points of L and $\phi_1(L)$ of $\mu(p)=k$,

$$C_k := \bigoplus_{\mu(p)=k}\mathbf{Z}_2 p.$$

We define a linear map $\partial : C_k\to C_{k-1}$ in terms of the canonical bases by

$$\partial p := \sum_{\mu(q)=k-1}\sharp\hat{\mathcal{M}}(p,q)q,$$

where $\sharp\hat{\mathcal{M}}(p,q)$ counts the elements of $\hat{\mathcal{M}}(p,q)$ modulo 2. Then we can calculate

$$\begin{aligned}\partial\partial x &= \partial\sum_{\mu(y)=\mu(x)-1}\sharp\hat{\mathcal{M}}(x,y)y\\ &= \sum_{\mu(y)=\mu(x)-1}\sum_{\mu(z)=\mu(y)-1}\sharp\hat{\mathcal{M}}(x,y)\sharp\hat{\mathcal{M}}(y,z)z.\end{aligned}$$

From Theorem 4.6 (2)

$$\sum_{\mu(y)=\mu(x)-1} \sharp\hat{\mathcal{M}}(x,y)\sharp\hat{\mathcal{M}}(y,z)$$

is zero modulo 2.

Theorem 4.7 $\partial^2 = 0$.

We call the homology *Floer homology* for Lagrangian submanifolds L and $\phi_1(L)$. Although we used J_t to construct Floer homology, the following theorem holds.

Theorem 4.8 *There is an isomorphism of vector spaces between Floer homologies for generic J_t and J_t'. Moreover the Floer homologies for $(L, \phi_1(L))$ and $(L, \phi_1'(L))$ are isomorphic.*

To calculate Floer homology we can choose convenient Hamiltonian isotopies from Theorem 4.8. If H_t is small enough, then $\phi_1(L)$ is in $N(L)$, where $N(L)$ is the neighborhood as in Theorem 3.1. Moreover we choose H_t so that we can identify $\phi_1(L)$ with the graph of dh in $N(0_L)$, where h is a Morse function. We shall use f to denote $-h$ in the following lemma.

Lemma 4.9 *We consider a metric on X which, on $N(L)$, is induced by a metric g on L, and also almost complex structures J on X which, on $N(L)$, maps the vertical tangent vectors to horizontal tangent vectors with respect to the Levi-Civita connection of g. If $x : \mathbf{R} \to L$ satisfies*

$$\frac{dx}{d\tau} = -\mathrm{grad} f,$$

then

$$\frac{\partial \overline{x}}{\partial \tau} + J_t(\overline{x}(\tau, t))\frac{\partial \overline{x}}{\partial t} = 0,$$

where $\overline{x}(\tau, t) := \phi_t(x(\tau))$ and $J_t := \phi_{t*} J \phi_{t*}^{-1}$.

We can identify $L \cap \phi_1(L)$ with the set of critical points of f and the boundary operator of Floer homology with the one of the Morse homology over $\mathbf{Z}_2$.

Theorem 4.10 *The Floer homology for L and $\phi_1(L)$ is isomorphic to the singular homology of L over $\mathbf{Z}_2$.*

Hence

$$\begin{aligned}
&\sharp\{L \cap \phi_1(L)\} \\
&\quad = \text{ the number of the generators of the Floer's chain complex for } L \text{ and } \phi_1(L) \\
&\quad \geq \text{ the rank of the Floer homology for } L \text{ and } \phi_1(L) \\
&\quad = \text{ the rank of the Morse homology of } L \text{ over } \mathbf{Z}_2 \\
&\quad = \text{ the rank of the singular homology of } L \text{ over } \mathbf{Z}_2.
\end{aligned}$$

This completes the proof of Theorem 4.1.

References

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