Energy of knots and conformal geometry

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Just like a minimal surface is modeled on the "optimal surface" of a soap film with a given boundary curve, one can ask whether we can define an "*optimal knot*", a beautiful knot which represents its knot type. Energy of knots was introduced for this purpose. The basic philosophy due to Fukuhara and Sakuma independently is as follows.

Suppose there is a non-conductive knotted string which is charged uniformly in a non-conductive viscous fluid. Then it might evolve itself to decrease its electrostatic energy without intersecting itself because of Coulomb's repulsive force until it comes to a critical point of the energy. Then we might be able to define an "optimal embedding" of a knot by an *energy minimizer*, which is an embedding that attains the minimum energy within its isotopy class. Thus our motivational problem can be stated as:

Can we define an *energy* for knots and "singular knots" with double points so that the energy is infinite for singular knots that are not embeddings, and so that the gradient flow of the energy evolvs a knot to the "optimal embedding" while preserving its isotopy type?

The first example of such an energy, $E_\circ^{(2)}$, was defined as the renormalization of a "modified" electrostatic energy of uniformly charged knots so that the magnitude $|\boldsymbol{F}(x,y)|$ of Coulomb's repulsive force between a pair of unit point charges of distance $r=|x-y|$ satisfies $|\boldsymbol{F}(x,y)|\propto r^{-3}$ ([**?**]):

$$\begin{aligned} E_\circ^{(2)}(K) &= \lim_{\epsilon\to 0}\left\{\iint_{(x,y)\in K\times K, d_K(x,y)\geq\epsilon}\frac{dxdy}{|x-y|^2}-\frac{2}{\epsilon}\right\} \\ &= -4+\iint_{K\times K}\quad\frac{1}{|x-y|^2}-\frac{1}{d_K(x,y)^2}\quad dxdy\,, \end{aligned}$$

where, $d_K(x,y)$ denotes the (shorter) arclength between x and y along K.

The answer to the above question is that there is an energy minimizer in any *prime* knot type ([**?**]), although it is conjectured through numerical experiments that there would be no energy minimizers in any composite knot type $[K_1\sharp K_2]$, because both tangles representing $[K_1]$ and $[K_2]$ pull tight to points if the knot evolves itself to decrease its energy ([**?**]).

This energy $E_\circ^{(2)}$ was proved to be invariant under Möbius transformations of $\mathbb{R}^3 \cup \{\infty\}$ by Freedman, He, and Wang ([**?**]). In the joint work with R. Langevin ([**?**]), we introduced a conformal geometric interpretation.

Let $\Sigma_K(x,y)$ denote the 2-sphere which is tangent to the knot K both at x and at y. Let us identify $\Sigma_K(x,y)$ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$ through a stereographic projection p. Then the four points $x, x+dx, y$, and $y+dy$ on $\Sigma_K(x,y)$ can be considered as four complex numbers $\tilde{x} = p(x)$, $\tilde{x}+\widetilde{dx} = p(x+dx)$, $\tilde{y} = p(y)$, and $\tilde{y}+\widetilde{dy} = p(y+dy)$. Let $\Omega_K(x,y)$ be the cross ratio $(\tilde{x}+\widetilde{dx}, \tilde{y}; \tilde{x}, \tilde{y}+\widetilde{dy})$:

$$\Omega_K(x,y) = \frac{(\tilde{x}+\widetilde{dx}) - \tilde{x}}{(\tilde{x}+\widetilde{dx}) - (\tilde{y}+\widetilde{dy})} : \frac{\tilde{y}-\tilde{x}}{\tilde{y}-(\tilde{y}+\widetilde{dy})} \sim \frac{\widetilde{dx}\widetilde{dy}}{(\tilde{x}-\tilde{y})^2}.$$

Then $\Omega_K(x,y)$ is independent of the choice of the stereographic projection p. We call $\Omega_K(x,y)$ the *infinitesimal cross ratio* of the knot K. It is a meromorphic 2-form on $K \times K$ with the pole of order 2 at the diagonal Δ. It is invariant under the diagonal action $(x,y) \mapsto (g \cdot x, g \cdot y)$ of the Möbius group to $K \times K \subset S^3 \times S^3$.

Then $E_\circ^{(2)}$ can be expressed as

$$E_\circ^{(2)}(K) = \iint_{K \times K \setminus \Delta} \{|\Omega_{CR}| - \mathfrak{Re}\, \Omega_{CR}\}.$$

This story has several generalizations and open problems: A different index of the integrand of the energy or a different metric of the ambient space will give a different answer to the question about the existence of the "optimal knots" as the energy minimizers. There are studies on the relation between energies and other geometric quantities to measure the complexity of knots such as average crossing number and thickness. Numerical experiments have been carried out. The infinitesimal cross ratio can express other conformally invariant knot energies, for example, we can consider the imaginary part of the infinitesimal. A higher dimensional anologue can be considered, and so on... But one of our dreams which is far now is to use an energy of knots as the Morse function on the space of knots.

References

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