Introduction to AdS-CFT

lectures by
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Abstract

These lectures present an introduction to AdS-CFT, and is intended both for begining and more advanced graduate students, which are familiar with quantum field theory and have a working knowledge of their basic methods. Familiarity with supersymmetry, general relativity and string theory is helpful, but not necessary, as the course intends to be as self-contained as possible. I will introduce the needed elements of field and gauge theory, general relativity, supersymmetry, supergravity, strings and conformal field theory. Then I describe the basic AdS-CFT scenario, of $\mathcal{N}=4$ Super Yang-Mills's relation to string theory in $AdS_5 \times S_5$, and applications that can be derived from it: 3-point functions, quark-antiquark potential, finite temperature and scattering processes, the pp wave correspondence and spin chains.

1 Elements of quantum field theory and gauge theory

Here I will review some elements of quantum field theory and gauge theory that will be needed in the following.

The Fenyman path integral and Feynman diagrams

Conventions: throughout this course, I will use theorist's conventions, where $\hbar = c = 1$. To reintroduce $\hbar$ and c one can use dimensional analysis. In this conventions, there is only one dimensionful unit, $mass = 1/length = energy = 1/time = ...$ and when I speak of dimension of a quantity I refer to mass dimension, i.e. the mass dimension of d^4x, $[d^4x]$, is -4. The Minkowski metric $\eta^{\mu\nu}$ will have signature $(-+++)$, thus $\eta^{\mu\nu} = diag(-1,+1,+1,+1)$.

I will use the example of the scalar field $\phi(x)$, that transforms as $\phi'(x') = \phi(x)$ under a coordinate transformation $x_\mu \to x'_\mu$. The action of such a field is of the type

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \tag{1.1}$$

where $\mathcal{L}$ is the Lagrangian density.

Classically, one varies this action with respect to $\phi(x)$ to give the classical equations of motion for $\phi(x)$

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \tag{1.2}$$

Quantum mechanically, the field $\phi(x)$ is not observable anymore, and instead one must use the vacuum expectation value (VEV) of the scalar field quantum operator instead, which is given as a "path integral"

$$< 0|\hat{\phi}(x_1)|0 >= \int \mathcal{D}\phi e^{iS[\phi]}\phi(x_1) \tag{1.3}$$

Here the symbol $\mathcal{D}\phi$ represents a discretization of spacetime followed by integration of the field at each discrete point:

$$\mathcal{D}\phi(x) = \prod_i \int d\phi(x_i) \tag{1.4}$$

A generalization of this object is the correlation function or n-point function

$$G_n(x_1, ..., x_n) =< 0|T\{\hat{\phi}(x_1)...\hat{\phi}(x_n)\}|0 > \tag{1.5}$$

The generating function of the correlation functions is called the partition function,

$$Z[J] = \int \mathcal{D}\phi e^{iS[\phi]+i\int d^4x J(x)\phi(x)} \tag{1.6}$$

It turns out to be convenient to write quantum field theory in Euclidean signature, and go between the Minkowski signature $(-+++)$ and the Euclidean signature $(++++)$ via a Wick rotation, $t = -it_E$ and $iS \to -S_E$, where t_E is Euclidean time (with positive metric) and S_E is the Euclidean action.

The partition function in Euclidean space is

$$Z_E[J] = \int \mathcal{D}\phi e^{-S_E[\phi]+\int d^4x J(x)\phi(x)} \tag{1.7}$$

and the correlation functions

$$G_n(x_1,...,x_n) = \int \mathcal{D}\phi e^{-S_E[\phi]}\phi(x_1)...\phi(x_n) \tag{1.8}$$

are given by differentiation of the partition function

$$G_n(x_1,...,x_n) = \frac{\delta}{\delta J(x_1)}...\frac{\delta}{\delta J(x_n)}\int \mathcal{D}\phi e^{-S_E[\phi]+\int d^4x J(x)\phi(x)}|_{J=0} \tag{1.9}$$

This formula can be calculated in perturbation theory, using the so called "Feynman diagrams". To exemplify it, we will use a scalar field Euclidean action

$$S_E[\phi] = \int d^4x[\frac{1}{2}(\partial_\mu\phi)^2 + m^2\phi^2 + V(\phi)] \tag{1.10}$$

Here I have used the notation

$$(\partial_\mu\phi)^2 = \partial_\mu\phi\partial^\mu\phi = \partial_\mu\phi\partial_\nu\phi\eta^{\mu\nu} = -\dot{\phi}^2 + (\vec{\nabla}\phi)^2 \tag{1.11}$$

Moreover, for concreteness, I will use $V = \lambda\phi^4$.

Then, the **Feynman diagram in x space** is obtained as follows. One draws a diagram, in the example in Fig.1a) it is the so-called "setting Sun" diagram.

A line between point x and point y represent the propagator

$$\Delta(x,y) = [-\partial_\mu\partial^\mu + m^2]^{-1} = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2+m^2} = \frac{1}{(x-y)^2} \tag{1.12}$$

A 4-vertex at point x represents the vertex

$$\int d^4x(-\lambda) \tag{1.13}$$

And then the value of the Feynman diagram, $F_D^{(N)}(x_1,...x_n)$ is obtained by multiplying all the above elements, and the value of the n-point function is obtained by summing over diagrams, and over the number of 4-vertices N with a weight factor:

$$G_n(x_1,...,x_n) = \sum_{N\geq 0}\frac{1}{N!}\sum_{diag\ D}F_D^{(N)}(x_1,...,x_n) \tag{1.14}$$

(Equivalently, one can use a $\lambda\phi^4/4!$ potential and construct only *topologically inequivalent* diagrams and the vertices are still $\int d^4x(-\lambda)$, but we now multiply each inequivalent diagram by a statistical weight factor).

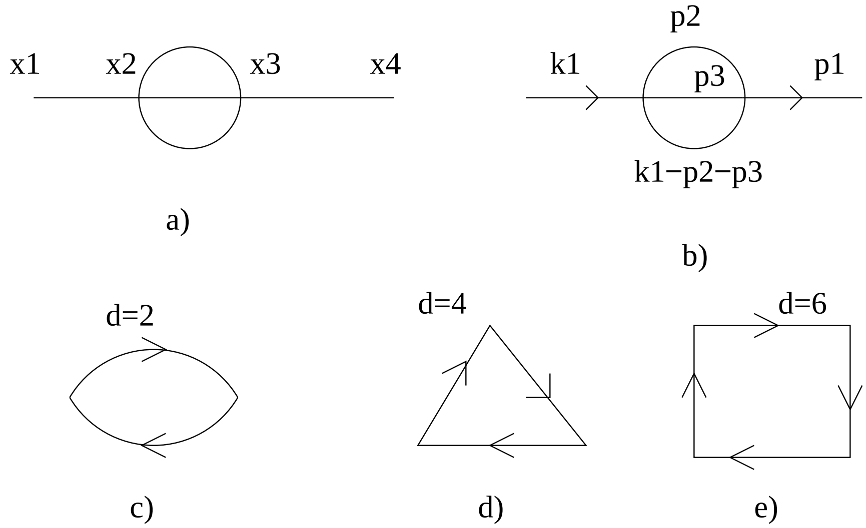

Figure 1: a) "Setting sun" diagram in x-space. b) "Setting sun" diagram in momentum space. c)anomalous diagram in 2 dimensions; d)anomalous diagram (triangle) in 4 dimensions; e)anomalous diagram (box) in 6 dimensions.

We mentioned that the VEV of the scalar field operator is an obsevable. In fact, the normalized VEV in the presence of a source $J(x)$,

$$\phi(x;J) = \frac{{}_J < 0|\hat{\phi}(x)|0 >_J}{{}_J < 0|0 >_J} = \frac{1}{Z[J]}\int \mathcal{D}\phi e^{-S[\phi]+J\cdot\phi}\phi(x) = \frac{\delta}{\delta J}\ln Z[J] \tag{1.15}$$

is called the classical field and satisfies an analog (quantum version) of the classical field equation.

S matrices

For real scattering, one constructs incoming and outgoing wavefunctions, representing actual states, in terms of the idealized states of fixed (external) momenta $\vec{k}$.

Then one treats the scattering of these idealized states and at the end one convolutes with the wavefunctions. The S matrix defines the transition amplitude between the idealized states by

$$< \vec{p}_1, \vec{p}_2, ...|S|\vec{k}_1, \vec{k}_2, ... > \tag{1.16}$$

The value of this S matrix transition amplitude is given in terms of Feynman diagrams in momentum space. The diagrams are of a restricted type: connected (doesn't contain disconnected pieces) and amputated (which means that one does not use propagators for the external lines).

For instance, the setting sun diagram with external momenta k_1 and p_1 and internal momenta p_2, p_3 and $k_1 - p_2 - p_3$ in Fig.1b) is

$$\delta^4(k_1 - p_1)\int d^4p_2 d^4p_3 \lambda^2 \frac{1}{p_2^2+m^2}\frac{1}{p_3^2+m^2}\frac{1}{(k_1-p_2-p_3)^2+m_4^2} \tag{1.17}$$

The **LSZ formulation** relates S matrices in Minkowski space with correlation functions as follows. The Fourier transformed $n+m$-point function near the physical poles $P_I^2 = M^2$ behaves as

$$\tilde{G}_{n+m}(p_1,...,p_n)(x_1,...,x_n) \sim \left(\prod_{i=1}^{n} \frac{\sqrt{Z}i}{p_i^2 - m^2 + i\epsilon}\right)\left(\prod_{j+1}^{m} \frac{\sqrt{Z}i}{k_j^2 - m^2 + i\epsilon}\right) < p_1,...,p_n|S|k_1,..,k_m > \tag{1.18}$$

For this reason, the study of correlation functions, which is easier, is preferred, since any physical process can be extracted from them as above.

If the external states are not states of a single field, but of a composite field $\mathcal{O}(x)$, e.g.

$$\mathcal{O}_{\mu\nu}(x) = (\partial_\mu\phi\partial_\nu\phi)(x)(+...) \tag{1.19}$$

it is useful to define Euclidean space correlation functions for these operators

$$\begin{aligned} < \mathcal{O}(x_1)...\mathcal{O}(x_n) >_{Eucl} &= \int \mathcal{D}\phi e^{-S_E}\mathcal{O}(x_1)...\mathcal{O}(x_n) \\ &= \frac{\delta^n}{\delta J(x_1)...\delta J(x_n)} \int \mathcal{D}\phi e^{-S_E + \int d^4x \mathcal{O}(x)J(x)}|_{J=0} \end{aligned} \tag{1.20}$$

which can be obtained from the generating functional

$$Z_{\mathcal{O}}[J] = \int \mathcal{D}\phi e^{-S_E + \int d^4x\mathcal{O}(x)J(x)} \tag{1.21}$$

Yang-Mills theory and gauge groups

Electromagnetism

In electromagnetism we have a gauge field

$$A_\mu(x) = (\phi(\vec{x},t), \vec{A}(\vec{x},t)) \tag{1.22}$$

with the field strength (containg the $\vec{E}$ and $\vec{B}$ fields)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = 2\partial_{[\mu}A_{\nu]} \tag{1.23}$$

The observables like $\vec{E}$ and $\vec{B}$ are defined in terms of $F_{\mu\nu}$ and as such the theory has a gauge symmetry under a U(1) group, that leaves $F_{\mu\nu}$ invariant

$$\delta A_\mu = \partial_\mu\lambda; \quad \delta F_{\mu\nu} = 2\partial_{[\mu}\partial_{\nu]}\lambda = 0 \tag{1.24}$$

The Minkowski space action is

$$S_{Mink} = -\frac{1}{4}\int d^4x F_{\mu\nu}^2 \tag{1.25}$$

which becomes in Euclidean space

$$S_E = \frac{1}{4}\int d^4x (F_{\mu\nu})^2 = \frac{1}{4}\int d^4x F_{\mu\nu}F_{\rho\sigma}\eta^{\mu\rho}\eta^{\nu\sigma} \tag{1.26}$$

The coupling of electromagnetism to a scalar field ϕ and a fermion field ϕ is obtained as follows

$$
\begin{aligned}
S_E^{total} &= S_{E,A} + \int d^4x[\bar{\psi}(\not{D}+m)\psi + (D_\mu\phi)^* D^\mu\phi] \\
\not{D} &= D_\mu\gamma^\mu; \quad D_\mu = \partial_\mu - ieA_\mu
\end{aligned}
\tag{1.27}
$$

This is known as the minimal coupling. Then there is a U(1) local symmetry that extends the above gauge symmetry, namely

$$
\psi' = e^{ie\lambda(x)}\psi; \quad \phi' = e^{ie\lambda(x)}\phi \tag{1.28}
$$

under which $D_\mu\psi$ transforms as $e^{ie\lambda}D_\mu\psi$, i.e transforms *covariantly*, as does $D_\mu\phi$.

The reverse is also possible, namely we can start with the action for ϕ and ψ only, with ∂_μ instead of D_μ. It will have the symmetry in (1.28), except with a global parameter only. If we want to promote the global symmetry to a local one, we need to introduce a gauge field and minimal coupling as above.

In the following, I will sometimes replace e by ie, thus $D_\mu = \partial_\mu + eA_\mu$.

Yang-Mills fields

Yang-Mills fields A_μ^a are self-interacting gauge fields, where a is an index belonging to a nonabelian gauge group. There is thus a 3-point self-interaction of the gauge fields $A_\mu^a, A_\nu^b, A_\rho^c$, that is defined by the constants $f^a{}_{bc}$.

The gauge group G has generators $(T^a)_{ij}$ in the representation R. T^a satisfy the Lie algebra of the group,

$$
[T_a, T_b] = f_{ab}{}^c T_c \tag{1.29}
$$

The group G is usually $SU(N), SO(N)$. The adjoint representation is defined by $(T^a)_{bc} = f^a{}_{bc}$. Then the gauge fields live in the adjoint representation and the field strength is

$$
F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^a{}_{bc}A_\mu^b A_\nu^c \tag{1.30}
$$

One can define $A = A^a T_a$ and $F_{\mu\nu} = F_{\mu\nu}^a T_a$ in terms of which we have

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu, A_\nu] \tag{1.31}
$$

(If one further defines the forms $F = 1/2F_{\mu\nu}dx^\mu \wedge dx^\nu$ and $A = A_\mu dx^\nu$ where wedge $\wedge$ denotes antisymmetrization, one has $F = dA + gA \wedge A$).

The generators T^a are taken to be antihermitian, their normalization being defined by their trace in the fundamental representation,

$$
trT^aT^b = -\frac{1}{2}\delta^{ab} \tag{1.32}
$$

and here group indices are raised and lowered with δ^{ab}.

The local symmetry under the group G or gauge symmetry has now the infinitesimal form

$$
\delta A_\mu^a = (D_\mu\epsilon)^a \tag{1.33}
$$

where

$$(D_\mu \epsilon)^a = \partial_\mu \epsilon^a + g f^a{}_{bc} A^b_\mu \epsilon^c \tag{1.34}$$

The finite form of the transformation is

$$A^U_\mu(x) = U^{-1}(x) A_\mu(x) U(x) + U^{-1}\partial_\mu U(x); \quad U = e^{\lambda^a T_a} = e^\lambda \tag{1.35}$$

and if $\lambda^a = \epsilon^a$ =small, we get the previous. This transformation leaves invariant the Euclidean action

$$S_E = -\frac{1}{2}\int d^4x tr(F_{\mu\nu}F^{\mu\nu}) = \frac{1}{4}\int d^4x F^a_{\mu\nu}F^{b,\mu\nu}\delta_{ab} \tag{1.36}$$

whereas the fields stregth transforms covariantly, i.e.

$$F'_{\mu\nu} = U^{-1}(x) F_{\mu\nu} U(x) \tag{1.37}$$

Coupling with other fields is done again by using the covariant derivative. In representation R, the covariant derivative D_μ (that also transforms covariantly) is

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu + g(T^a)_{ij} A^a_\mu(x) \tag{1.38}$$

and one replaces ∂_μ by D_μ, e.g. for a fermion, $\bar\psi \partial\!\!\!/ \psi \to \bar\psi D\!\!\!\!/ \psi$.

Symmetry currents and anomalies

The Noether theorem states that a global classical symmetry corresponds to a conserved current (on-shell), i.e.

$$\delta_{symm.}\mathcal{L} = \epsilon^a \partial_\mu j^{\mu,a} \tag{1.39}$$

so that a classical symmetry corresponds to having the Noether current $j^{\mu,a}$ conserved, i.e. $\partial_\mu j^{\mu,a}$=0. If the transformation is

$$\delta\phi^i = \epsilon^a (T^a)_{ij}\phi^j \tag{1.40}$$

then the Noether current is

$$j^{\mu,a} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi^i)} T^a_{ij}\phi^j \tag{1.41}$$

Quantum mechanically however, the current can have an *anomaly*, i.e. $< \partial_\mu j^{\mu,a} > \neq 0$. In momentum space, this will be $p_\mu < j^{\mu,a} > \neq 0$.

As an example, take the Lagrangian

$$\mathcal{L} = \bar\psi^i \gamma^\mu D_\mu \psi_i \tag{1.42}$$

with $\delta\psi^i = \epsilon^a (T^a)_{ij}\psi^j$. It gives the symmetry (Noether) current

$$j^{\mu,a} = \bar\psi^i \gamma^\mu T^a_{ij}\psi^j \tag{1.43}$$

Some observations can be made about this example. First, j^a_μ is a composite operator. Second, if ψ^i has also some gauge (local symmetry) indices, then $j^{\mu,a}$ is gauge invariant, so it can represent a physical state.

One can use the formalism for composite operators and define the correlator

$$< j^{\mu_1,a_1}(x_1),...j^{\mu_n,a_n}(x_n) >= \frac{\delta^n}{\delta A^{a_1}_{\mu_1}(x_1)...\delta A^{a_n}_{\nu_n}(x_n)} \int \mathcal{D}[fields] e^{-S+\int d^4x j^{\mu,a}(x) A^a_\mu(x)} \tag{1.44}$$

which will then be a correlator of some external physical states (observables).

We will see that this kind of correlators are obtained from AdS-CFT. The current anomaly can manifest itself also in this correlator. $j^{\mu,a}$ is inserted inside the quantum average, thus in momentum space, we could a priori have the anomaly

$$p_{\mu_1} < j^{\mu_1,a_1},...j^{\mu_n,a_n} >\neq 0 \tag{1.45}$$

In general, the anomaly is 1-loop only, and is given by polygon graphs, i.e. a 1-loop contribution (a 1-loop Feynman diagram) to an n-point current correlator that looks like a n-polygon with vertices= the $x_1, ..x_n$ points. In d=2, only the 2-point correlator is anomalous by the Feynman diagram in Fig.1c, in d=4, the 3-point, by a triangle Feynman diagram, as in Fig.1d, in d=6 the 4-point, by a box (square) graph, as in Fig.1e, etc.

Therefore, in d=4, the anomaly is called triangle anomaly.

Important concepts to remember

- Correlation functions are given by a Feynman diagram expansion and appear as derivatives of the partition function
- S matrices defining physical scatterings are obtained via the LSZ formalism from the poles of the correlation functions
- Correlation functions of composite operators are obtained from a partition function with sources coupling to the operators
- Coupling of fields to electromagnetism is done via minimal coupling, replacing the derivatives d with the covariant derivatives $D = d - ieA$.
- Yang-Mills fields are self-coupled. Both the covariant derivative and the field strength transform covariantly.
- Classically, the Noether theorem associates every symmetry with a conserved current.
- Quantum mechanically, global symmetries can have an anomaly, i.e the current is not conserved, when inserted inside a quantum average.
- The anomaly comes only from 1-loop Feynman diagrams. In d=4, it comes from a triangle, thus only affects the 3-point function.
- In a gauge theory, the current of a global symmetry is gauge invariant, thus corresponds to some physical state.

Exercises, Section 1

1. If we have the partition function

$$Z[J] = \exp\{-\int d^4x[(\int d^4x_0 K(x,x_0)J(x_0))(-\frac{\Box_x}{2})(\int d^4y_0 K(x,y_0)J(y_0))$$
$$+\lambda(\int d^4x_0 K(x,x_0)(J(x_0)))^3]\} \quad (1.46)$$

write an expression for $G_2(x,y)$ and $G_3(x,y)$.

2. If we have the Euclidean action

$$S_E = \int d^4x[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2\phi^2}{2} + \lambda\phi^3] \quad (1.47)$$

write down the integral for the Feynman diagram in Fig.2a.

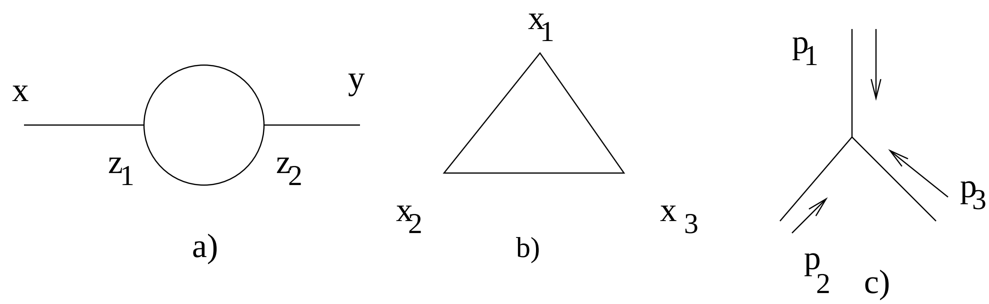

Figure 2: a) Setting sun diagram in x space; b) Triangle diagram in x space; c) Star diagram in p space

3. Show that the Fourier transform of the triangle diagram in x space in Fig.2b is the star diagram in p space in Fig.2c.

4. Derive the Hamiltonian $H(\vec{E},\vec{B})$ for the electromagnetic field by putting $A_0 = 0$, from $S_M = -\int F_{\mu\nu}^2/4$.

5. Show that $F_{\mu\nu} = [D_\mu, D_\nu]/g$. What is the infinitesimal transformation of $F_{\mu\nu}$? For SO(d) groups, the adjoint representation is antisymmetric, (ab). Calculate $f^{(ab)}{}_{(cd)(ef)}$ and write down $F_{\mu\nu}^{ab}$.

6. Consider the action

$$S = -\frac{1}{4}\int F_{\mu\nu}^2 + \frac{1}{2}\int \bar{\psi}(\not{D}+m)\psi + \frac{1}{2}\int (D_\mu\phi)^2 D^\mu\phi \quad (1.48)$$

and the U(1) electromagnetic transformation. Calculate the Noether current.

2 Basics of general relativity; Anti de Sitter space.

Curved spacetime and geometry

In **special relativity**, one postulates that the speed of light is constant in all inertial reference frames, i.e. $c = 1$. As a result, the line element

$$ds^2 = -dt^2 + d\vec{x}^2 = \eta_{ij} dx^i dx^j \tag{2.1}$$

is invariant, and is called the invariant distance. Here $\eta_{ij} = diag(-1, 1, ..., 1)$. Therefore the symmetry group of general relativity is the group that leaves the above line element invariant, namely SO(1,3), or in general SO(1,d-1).

This *Lorentz group* is a generalized rotation group: The rotation group SO(3) is the group that leaves the 3 dimensional length $d\vec{x}^2$ invariant. The Lorentz transformation is a generalized rotation

$$x'^i = \Lambda^i{}_j x^j; \quad \Lambda^i{}_j \in SO(1,3) \tag{2.2}$$

Therefore the statement of special relativity is that physics is Lorentz invariant (invariant under the Lorentz group SO(1,3) of generalized rotations).

In **general relativity**, one considers a more general spacetime, specifically a curved spacetime, defined by the distance between two points, or line element,

$$ds^2 = g_{ij}(x) dx^i dx^j \tag{2.3}$$

where $g_{ij}(x)$ are arbitrary functions called *the metric* (sometimes one refers to ds^2 as the metric). This situation is depicted in Fig.3a.

As we can see from the definition, the metric $g_{ij}(x)$ is a symmetric matrix.

To understand this, let us take the example of the sphere, specifically the familiar example of a 2-sphere embedded in 3 dimensional space. Then the metric in the embedding space is the usual Euclidean distance

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^3 \tag{2.4}$$

but if we are on a two-sphere we have the constraint

$$\begin{aligned} & x_1^2 + x_2^2 + x_3^2 = R^2 \Rightarrow 2(x_1 dx_1 + x_2 dx_2 + x_3 dx_3) = 0 \\ & \Rightarrow dx_3 = -\frac{x_1 dx_1 + x_2 dx_2}{x_3} = -\frac{x_1}{\sqrt{R^2 - x_1^2 - x_2^2}} dx_1 - \frac{x_2}{\sqrt{R^2 - x_1^2 - x_2^2}} dx_2 \end{aligned} \tag{2.5}$$

which therefore gives the induced metric (line element) on the sphere

$$ds^2 = dx_1^2 (1 + \frac{x_1^2}{R^2 - x_1^2 - x_2^2}) + dx_2^2 (1 + \frac{x_2^2}{R^2 - x_1^2 - x_2^2}) + 2dx_1 dx_2 \frac{x_1 x_2}{R^2 - x_1^2 - x_2^2} = g_{ij} dx^i dx^j \tag{2.6}$$

So this is an example of a curved d-dimensional space which is obtained by emebedding it into a flat (Euclidean or Minkowski) d+1 dimensional space. But if the metric $g_{ij}(x)$ are arbitrary functions, then one cannot in general embed such a space in flat d+1 dimensional space: there are $d(d+1)/2$ functions $g_{ij}(x)$ to be obtained and only one function (in the above

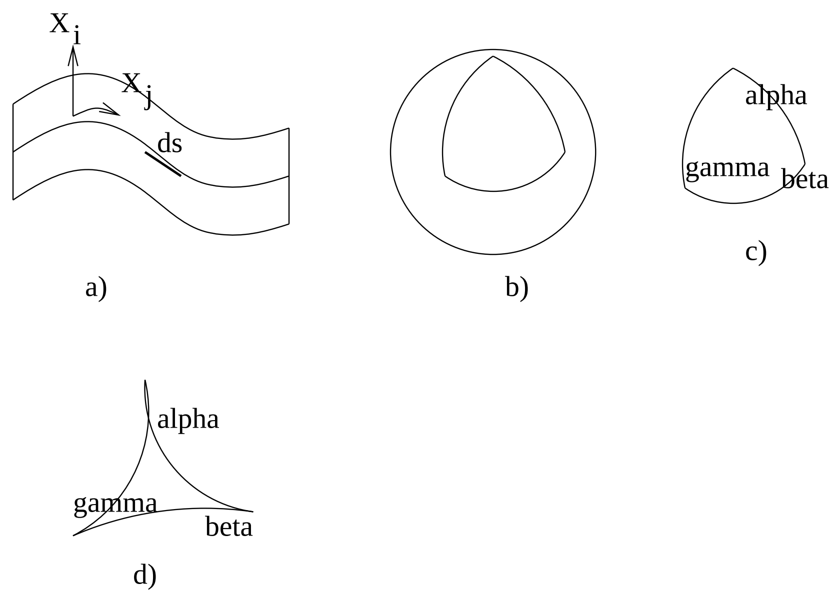

Figure 3: a) curved space. The functional form of the distance between 2 points depends on local coordinates. b) A triangle on a sphere, made from two meridian lines and a seqment of the equator has two angles of 90^0 $(\pi/2)$. c) The same triangle, drawn for a general curved space of positive curvature, emphasizing that the sum of the angles of the triangle exceeds 180^0 (π). d) In a space of negative curvature, the sum of the angles of the triangle is below 180^0 (π).

example, the function $x_3(x_1,x_2)$), together with d coordinate transformations $x'_i = x'_i(x_j)$ available for the embedding. In fact, we will see that the problem is even more complicated in general due to the signature of the metric (signs on the diagonal of the diagonalized matrix g_{ij}). Thus, even though a 2 dimensional metric has 3 components, equal to the 3 functions available for a 3 dimensional embedding, to embed a metric of Euclidean signature in 3d one needs to consider both 3d Euclidean and 3d Minkowski space.

That means that a general space can be *intrinsically curved*, defined not by embedding in a flat space, but by the arbitrary functions $g_{ij}(x)$ (the metric). In a general space, we define the *geodesic* as the line of shortest distance $\int_a^b ds$ between two points a and b.

In a curved space, the triangle made by 3 geodesics has an unusual property: the sum of the angles of the triangle, $\alpha+\beta+\gamma$ is not equal to π. For example, if we make a triangle from geodesics on the sphere as in Fig.3b, we can easily convince ourselves that $\alpha+\beta+\gamma > \pi$. In fact, by taking a vertex on the North Pole and two vertices on the Equator, we get $\beta = \gamma = \pi/2$ and $\alpha > 0$. This is the situation for a space with positive curvature, $R > 0$: two parallel geodesics converge to a point (Fig.3c). In the example given, the two parallel geodesics are the lines between the North Pole and the Equator: both lines are perpendicular to the equator, therefore are parallel by definition, yet they converge at the North Pole.

But one can have also a space with negative curvature, $R < 0$, for which $\alpha+\beta+\gamma < \pi$ and two parallel geodesics diverge, as in Fig.3d. Such a space is for instance the so-called *Lobachevski space*, which is a two dimensional space of Euclidean signature (like the two dimensional sphere), i.e. the diagonalized metric has positive numbers on the diagonal. However, this metric cannot be obtained as an embedding in a Euclidean 3d space, but rather an embedding in a Minkowski 3 dimensional space, by

$$ds^2 = dx^2 + dy^2 - dz^2; \quad x^2 + y^2 - z^2 = -R^2 \tag{2.7}$$

Einstein's theory of general relativity makes two physical assumptions

- gravity is geometry: matter follows geodesics in a curved space, and the resulting motion appears to us as the effect of gravity. AND
- matter sources gravity: matter curves space, i.e. the source of spacetime curvature (and thus of gravity) is a matter distribution.

We can translate these assumptions into two mathematically well defined physical principles and an equation for the dynamics of gravity (Einstein's equation). The physical principles are

- Physics is invariant under general coordinate transformations

$$x'_i = x'_i(x_j) \Rightarrow ds^2 = g_{ij}(x)dx^i dx^j = g'_{ij}(x')dx'^i dx'^j \tag{2.8}$$

- The Equivalence principle, which can be stated as "there is no difference between acceleration and gravity" OR "if you are in a free falling elevator you cannot distinguish it from being weightless (without gravity)". This is only a *local* statement: for example, if you are falling towards a black hole, tidal forces will pull you apart before you reach it (gravity acts slightly differently at different points). The quantitative way to write this principle is

$$m_i = m_g \text{ where } \vec{F} = m_i\vec{a} \text{ (Newton's law) and } \vec{F}_g = m_g\vec{g} \text{ (gravitational force)} \tag{2.9}$$

In other words, both gravity and acceleration are manifestations of the curvature of space.

Before describing the dynamics of gravity (Einstein's equation), we must define the kinematics (objects used to describe gravity).

As we saw, the metric $g_{\mu\nu}$ changes when we make a coordinate transformation, thus different metrics can describe the same space. In fact, since the metric is symmetric, it has $d(d+1)/2$ components. But there are d coordinate transformations $x'_\mu(x_\nu)$ one can make that leave the physics invariant, thus we have $d(d-1)/2$ degrees of freedom that describe the curvature of space (different physics).

We need other objects besides the metric that can describe the space in a more invariant manner. The basic such object is called the Riemann tensor, $R^\mu{}_{\nu\rho\sigma}$. To define it, we first define the inverse metric, $g^{\mu\nu} = (g^{-1})_{\mu\nu}$ (matrix inverse), i.e. $g_{\mu\rho}g^{\rho\sigma} = \delta^\sigma_\mu$. Then we define an object that plays the role of "gauge field of gravity", the Christoffel symbol

$$\Gamma^\mu{}_{\nu\rho} = \frac{1}{2}g^{\mu\sigma}(\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\sigma\rho} - \partial_\sigma g_{\nu\rho}) \tag{2.10}$$

Then the Riemann tensor is like the "field strength of the gravity gauge field", in that its definition can be written as to mimic the definition of the field strength of an SO(n) gauge group,

$$F^{ab}_{\mu\nu} = \partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu + A^{ac}_\mu A^{cb}_\nu - A^{ac}_\nu A^{cb}_\mu \tag{2.11}$$

where a, b, c are fundamental SO(n) indices, i.e. ab (antisymmetric) is an adjoint index. We put brackets in the definition of the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ to emphasize the similarity with the above:

$$(R^\mu{}_\nu)_{\rho\sigma}(\Gamma) = \partial_\rho(\Gamma^\mu{}_\nu)_\sigma - \partial_\sigma(\Gamma^\mu{}_\nu)_\rho + (\Gamma^\mu{}_\lambda)_\rho(\Gamma^\lambda{}_\nu)_\sigma - (\Gamma^\mu{}_\lambda)_\sigma(\Gamma^\lambda{}_\nu)_\rho \tag{2.12}$$

the only difference being that here both "gauge" and "spacetime" indices are the same.

From the Riemann tensor we construct by contraction the Ricci tensor

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} \tag{2.13}$$

and the Ricci scalar $R = R_{\mu\nu}g^{\mu\nu}$. The Ricci scalar is coordinate invariant, so it is truly an invariant measure of the curvature of space. The Riemann and Ricci tensors are examples of tensors. A contravariant tensor A^μ transforms as dx^μ,

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}A^\nu \tag{2.14}$$

whereas a covariant tensor B_μ transforms as $\partial/\partial x^\mu$, i.e.

$$B'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}B_\nu \tag{2.15}$$

and a general tensor transforms as the product of the transformations of the indices. The metric $g_{\mu\nu}$, the Riemann $R^{\mu}{}_{\nu\rho\sigma}$ and Ricci $R_{\mu\nu}$ and R are tensors, but the Christoffel symbol $\Gamma^{\mu}{}_{\nu\rho}$ is not.

To describe physics in curved space, we replace the Lorentz metric $\eta_{\mu\nu}$ by the general metric $g_{\mu\nu}$, and Lorentz tensors with general tensors. One important observation is that ∂_μ is not a tensor! The tensor that replaces it is the curved space covariant derivative, D_μ,

$$D_\mu T^\rho_\nu \equiv \partial_\mu T^\rho_\nu + \Gamma^\rho{}_{\mu\sigma} T^\sigma_\nu - \Gamma^\sigma{}_{\mu\nu} T^\rho_\sigma \tag{2.16}$$

We are now ready to describe the dynamics of gravity, in the form of Einstein's equation. It is obtained by postulating an action for gravity. The invariant volume of integration over space is not d^dx anymore as in Minkowski or Euclidean space, but $d^dx\sqrt{-g} \equiv d^dx\sqrt{-\det(g_{\mu\nu})}$ (where the $-$ sign comes from the Minkowski signature of the metric). The Lagrangian has to be invariant under general coordinate transformations, thus it must be a scalar (tensor with no indices). There would be several possible choices for such a scalar, but the simplest possible one, the Ricci scalar, turns out to be correct (i.e. compatible with experiment). Thus, one postulates the Einstein-Hilbert action for gravity

$$S_{gravity} = -\frac{1}{16\pi G}\int d^dx\sqrt{-g}R \tag{2.17}$$

The equations of motion of this action are

$$\frac{\delta S_{grav}}{\delta g^{\mu\nu}} = 0: \quad R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{2.18}$$

and as we mentioned, this action is not fixed, it just happens to agree well with experiments. In fact, in quantum gravity/string theory, S_g could have quantum corrections of different functional form.

The next step is to put matter in curved space, since one of the physical principles was that matter sources gravity. This follows the above mentioned rules. For instance, the kinetic term for a scalar field in Minkowski space was

$$S_\phi = \frac{1}{2}\int d^4x(\partial_\mu\phi)(\partial_\nu\phi)\eta^{\mu\nu} \tag{2.19}$$

and it becomes now

$$\frac{1}{2}\int d^4x\sqrt{-g}(D_\mu\phi)(D_\nu\phi)g^{\mu\nu} = \frac{1}{2}\int d^4x\sqrt{-g}(\partial_\mu\phi)(\partial_\nu\phi)g^{\mu\nu} \tag{2.20}$$

where the last equality, of the partial derivative with the covariant derivative, is only valid for a scalar field. In general, we will have covariant derivatives in the action.

The variation of the matter action gives the energy-momentum tensor (known from electromagnetism). By definition, we have

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_{matter}}{\delta g^{\mu\nu}} \tag{2.21}$$

Then the sum of the gravity and matter action give the equation of motion

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{2.22}$$

known as the Einstein's equation. For a scalar field, we have

$$T^{\phi}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (\partial_\rho \phi)^2 \tag{2.23}$$

Global Structure: Penrose diagrams

Spaces of interest are infinite in extent, but have complicated topological and causal structure. To make sense of them, we use the Penrose diagrams. These are diagrams that preserve the causal and topological structure of space, and have infinity at a finite distance on the diagram.

To construct a Penrose diagram, we note that light propagates along $ds^2 = 0$, thus an overall factor ("conformal factor") in ds^2 is irrelevant. So we make coordinate transformations that bring infinity to a finite distance, and drop the conformal factors. For convenience, we usually get some type of flat space at the end of the calculation. Then, in the diagram, light rays are at 45 degrees ($\delta x = \delta t$ for light, in the final coordinates).

As an example, we draw the Penrose diagram of 2 dimensional Minkowski space,

$$ds^2 = -dt^2 + dx^2 \tag{2.24}$$

where $-\infty < t, x < +\infty$. We first make a transformation to "lightcone coordinates"

$$u_\pm = t \pm x \Rightarrow ds^2 = -du_+ du_- \tag{2.25}$$

followed by a transformation of the lightcone coordinates that makes them finite,

$$u_\pm = \tan \tilde{u}_\pm; \quad \tilde{u}_\pm = \frac{\tau \pm \theta}{2} \tag{2.26}$$

where the last transformation goes back to space-like and time-like coordinates θ and τ. Now the metric is

$$ds^2 = \frac{1}{4\cos^2 \tilde{u}_+ \cos^2 \tilde{u}_-} (-d\tau^2 + d\theta^2) \tag{2.27}$$

and by dropping the overall (conformal) factor we get back a flat two dimensional space, but now of finite extent. Indeed, we have that $|\tilde{u}_\pm| \leq \pi/2$, thus $|\tau \pm \theta| \leq \pi$, so the Penrose diagram is a diamond (a rotated square), as in Fig.4a)

For 3 dimensional Minkowski space the metric is again

$$ds^2 = -dt^2 + dr^2 (+r^2 d\theta^2) \tag{2.28}$$

and by dropping the angular dependence we get the same metric with as before, just that $r > 0$ now. So everything follows in the same way, just that $\theta > 0$ in the final form. Thus for 3d (and higher) Minkowski space, the Penrose diagram is a triangle (the $\tau > 0$ half of the 2d Penrose diagram), as in Fig.4b.

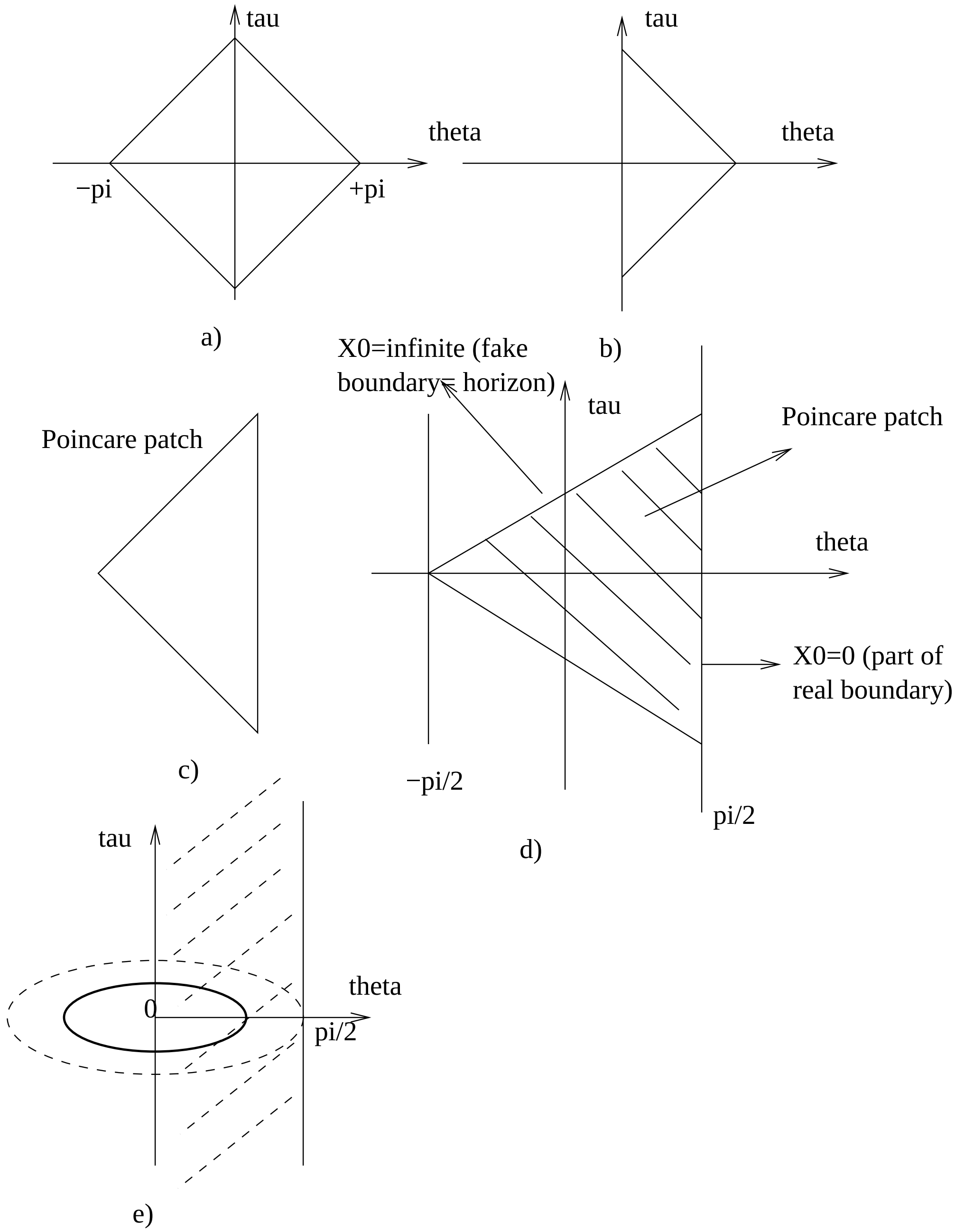

Figure 4: Penrose diagrams. a) Penrose diagram of 2 dimensional Minkowski space. b) Penrose diagram of 3 dimensional Minkowski space. c) Penrose diagram of the Poincare patch of Anti de Sitter space. d) Penrose diagram of global AdS_2 (2 dimensional Anti de Sitter), with the Poincare patch emphasized; $x_0 = 0$ is part of the boundary, but $x_0 = \infty$ is a fake boundary (horizon). e) Penrose diagram of global AdS_d for $d \geq 2$. It is half the Penrose diagram of AdS_2 rotated around the $\theta = 0$ axis.

Anti de Sitter space

Anti de Sitter space is a space of Lorentzian signature $(- + +...+)$, but of constant *negative* curvature. Thus is an analog of the Lobachevski space, which was a space of Euclidean signature and of constant negative curvature.

The anti in Anti de Sitter is because de Sitter space is defined as the space of Lorentzian signature and of constant positive curvature, thus an analog of the sphere (the sphere is the space of Euclidean signature and constant positive curvature).

In d dimensions, de Sitter space is defined by a sphere-like embedding in d+1 dimensions

$$\begin{aligned} ds^2 &= -dx_0^2 + \sum_{i=1}^{d-1} dx_i^2 + dx_{d+1}^2 \\ -x_0^2 &+ \sum_{i=1}^{d-1} x_i^2 + x_{d+1}^2 = R^2 \end{aligned} \tag{2.29}$$

thus as mentioned, this is the Lorentzian version of the sphere, and it is clearly invariant under the group SO(1,d) (the d dimensional sphere would be invariant under SO(d+1) rotating the d+1 embedding coordinates).

Similarly, in d dimensions, Anti de Sitter space is defined by a Lobachevski-like embedding in d+1 dimensions

$$\begin{aligned} ds^2 &= -dx_0^2 + \sum_{i=1}^{d-1} dx_i^2 - dx_{d+1}^2 \\ -x_0^2 &+ \sum_{i=1}^{d-1} x_i^2 - x_{d+1}^2 = -R^2 \end{aligned} \tag{2.30}$$

and is therefore the Lorentzian version of Lobachevski space. It is invariant under the group SO(2,d-1) that rotates the coordinates $x_\mu = (x_0, x_{d+1}, x_1, ..., x_{d-1})$ by $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

The metric of this space can be written in different forms, corresponding to different coordinate systems. In Poincare coordinates,

$$ds^2 = \frac{R^2}{x_0^2}(-dt^2 + \sum_{i=1}^{d-2} dx_i^2 + dx_0^2) \tag{2.31}$$

where $-\infty < t, x_i < +\infty$, but $0 < x_0 < +\infty$. Up to a conformal factor therefore, this is just like (flat) 3d Minkowski space, thus its Penrose diagram is the same, a triangle, as in Fig.4c. However, one now discovers that one does not cover all of the space! In the finite coordinates τ, θ, one finds that one can now analytically continue past the diagonal boundaries (there is no obstruction to doing so).

In these Poincare coordinates, we can understand Anti de Sitter space as a d-1 dimensional Minkowski space in $(t, x_1, ...x_{d-2})$ coordinates, with a "warp factor" (gravitational potential) that depends only on the additional coordinate x_0.

A coordinate system that does cover the whole space is called the global coordinates, and it gives the metric

$$ds_d^2 = R^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\vec{\Omega}_{d-2}^2) \tag{2.32}$$

where $d\vec{\Omega}^2$ is the metric on the unit sphere. This metric is written in a suggestive form, since the metric on the d-dimensional sphere can be written in a similar way,

$$ds_d^2 = R^2(\cos^2\rho dw^2 + d\rho^2 + \sin^2\rho d\vec{\Omega}_{d-2}^2) \tag{2.33}$$

The change of coordinates $\tan\theta = \sinh\rho$ gives the metric

$$ds_d^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta d\vec{\Omega}_{d-2}^2) \tag{2.34}$$

where $0 \leq \theta \leq \pi/2$ in all dimensions except 2, (where $-\pi/2 \leq \theta \leq \pi/2$), and τ is arbitrary, and from it we infer the Penrose diagram of global AdS_2 space (Anti de Sitter space in 2 dimensions) which is an infinite strip between $\theta = -\pi/2$ and $\theta = +\pi/2$. The "Poincare patch" covered by the Poincare coordinates, is a triangle region of it, with its vertical boundary being a segment of the infinite vertical boundary of the global Penrose diagram, as in Fig.4d.

The Penrose diagram of AdS_d is similar, but it is a cylinder obtained by the revolution of the infinite strip between $\theta = 0$ and $\theta = \pi/2$ around the $\theta = 0$ axis, as in Fig.4e. The "circle" of the revolution represents in fact a d-2 dimensional sphere. Thefore the boundary of AdS_d (d dimensional Anti de Sitter space) is $\mathbf{R}_\tau \times S_{d-2}$, the infinite vertical line of time times a d-2 dimensional sphere. This will be important in defining AdS-CFT correctly.

Finally, let us mention that Anti de Sitter space is a solution of the Einstein equation with a constant energy-momentum tensor, known as a *cosmological constant*, thus $T_{\mu\nu} = 2\Lambda g_{\mu\nu}$, coming from a constant term in the action, $\int d^4x\sqrt{-g}\Lambda$, so the Einstein equation is

$$R_{\mu\nu} - \frac{1}{2}g_\mu R = 16\pi G\Lambda g_{\mu\nu} \tag{2.35}$$

Important concepts to remember

- In general relativity, space is intrinsically curved
- In general relativity, physics is invariant under general coordinate transformations
- Gravity is the same as curvature of space, or gravity = local acceleration.
- The Christoffel symbol acts like a gauge field of gravity, giving the covariant derivative
- Its field strength is the Riemann tensor, whose scalar contraction, the Ricci scalar, is an invariant measure of curvature
- One postulates the action for gravity as $(1-/(16\pi G))\int\sqrt{-g}R$, giving Einstein's equations

- To understand the causal and topological structure of curved spaces, we draw Penrose diagrams, which bring infinity to a finite distance in a controlled way.

- de Sitter space is the Lorentzian signature version of the sphere; Anti de Sitter space is the Lorentzian version of Lobachevski space, a space of negative curvature.

- Anti de Sitter space in d dimensions has $SO(2, d-1)$ invariance.

- The Poincare coordinates only cover part of Anti de Sitter space, despite having maximum possible range (over the whole real line).

- Anti de Sitter space has a cosmological constant.

Exercises, section 2

1) Parallel the derivation in the text to find the metric on the 2-sphere in its usual form,

$$ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.36}$$

from the 3d Euclidean metric.

2) Show that on-shell, the graviton has degrees of freedom corresponding to a transverse (d-2 indices) symmetric traceless tensor.

3) Show that the metric $g_{\mu\nu}$ is covariantly constant ($D_\mu g_{\nu\rho} = 0$) by substituting the Christoffel symbols.

4) The Christoffel symbol $\Gamma^\mu_{\nu\rho}$ is not a tensor, and can be put to zero at any point by a choice of coordinates (Riemann normal coordinates, for instance), but $\delta\Gamma^\mu_{\nu\rho}$ is a tensor. Show that the variation of the Ricci scalar can be written as

$$\delta R = \delta^\rho_\mu g^{\nu\sigma}(\partial_\rho \delta\Gamma^\mu_{\nu\sigma} - \partial_\sigma \delta\Gamma^\mu_{\nu\rho}) + R_{\nu\sigma}\delta g^{\nu\sigma} \tag{2.37}$$

5) Parallel the calculation in 2d to show that the Penrose diagram of 3d Minkwoski space, with an angle ($0 \leq \phi \leq 2\pi$) supressed, is a triangle.

6) Substitute the coordinate transformation

$$X_0 = R\cosh\rho\cos\tau; \quad X_i = R\sinh\rho\Omega_i; \quad X_{d+1} = R\cosh\rho\sin\tau \tag{2.38}$$

to find the global metric of AdS space from the embedding (2,d-1) signature flat space.

3 Basics of supersymmetry

In the 1960's people were asking what kind of symmetries are possible in particle physics?

We know the Poincare symmetry defined by the Lorentz generators J_{ab} of the SO(1,3) Lorentz group and the generators of 3+1 dimensional translation symmetries, P_a.

We also know there are possible internal symmetries T_r of particle physics, such as the local U(1) of electromagnetism, the local SU(3) of QCD or the global SU(2) of isospin. These generators will form a Lie algebra

$$[T_r, T_s] = f_{rs}{}^t T_t \tag{3.1}$$

So the question arose: can they be combined, i.e. $[T_s, P_a] \neq 0, [T_s, J_{ab}] \neq 0$, such that maybe we could embed the SU(2) of isospin together with the SU(2) of spin into a larger group?

The answer turned out to be NO, in the form of the Coleman-Mandula theorem, which says that if the Poincare and internal symmetries were to combine, the S matrices for all processes would be zero.

But like all theorems, it was only as strong as its assumptions, and one of them was that the final algebra is a Lie algebra.

But people realized that one can generalize the notion of Lie algebra to a *graded Lie algebra* and thus evade the theorem. A graded Lie algebra is an algebra that has some generators Q^i_α that satisfy not a commuting law, but an anticommuting law

$$\{Q^i_\alpha, Q^j_\beta\} = \text{other generators} \tag{3.2}$$

Then the generators P_a, J_{ab} and T_r are called "even generators" and the Q^i_α are called "odd" generators. The graded Lie algebra then is of the type

$$[\text{even}, \ \text{even}] = \text{even}; \ \ \{\text{odd}, \ \text{odd}\} = \text{even}; \ \ [\text{even}, \ \text{odd}] = \text{odd} \tag{3.3}$$

So such a graded Lie algebra generalization of the Poincare + internal symmetries is possible. But what kind of symmetry would a Q^i_α generator describe?

$$[Q^i_\alpha, J_{ab}] = (...)J_{cd} \tag{3.4}$$

means that Q^i_α must be in a representation of J_{ab} (the Lorentz group). Because of the anticommuting nature of Q^i_α we choose the spinor representation. But a spinor field times a boson field gives a spinor field. Therefore when acting with Q^i_α (spinor) on a boson field, we will get a spinor field.

Therefore Q^i_α *gives a symmetry between bosons and fermions, called* **supersymmetry**!

$$\delta\, boson = fermion; \quad \delta\, fermion = boson \tag{3.5}$$

$\{Q_\alpha, Q_\beta\}$ is called the supersymmetry algebra, and the above graded Lie algebra is called the superalgebra.

Here Q^i_α is a spinor, with α a spinor index and i a label, thus the parameter of the transformation law, ϵ^i_α is a spinor also.

But what kind of spinor? In particle physics, Weyl spinors are used, that satisfy $\gamma_5\psi = \pm\psi$, but in supersymmetry one uses Majorana spinors, that satisfy the reality condition

$$\chi^C \equiv \chi^T C = \bar{\chi} \equiv \chi^\dagger i\gamma^0 \tag{3.6}$$

where C is the "charge conjugation matrix", that relates γ_m with γ_m^T. In 4 dimensions, it satisfies

$$C^T = -C; \quad C\gamma^m C^{-1} = -(\gamma^m)^T \tag{3.7}$$

And C is used to raise and lower indices, but since it is antisymmetric, one must define a convention for contraction of indices (the order matters).

The reason we use Majorana spinors is convenience, since it is easier to prove various supersymmetry identities, and then in the Lagrangian we can always go from a Majorana to a Weyl spinor and viceversa.

2 dimensional Wess Zumino model

We will exemplify supersymmetry with the simplest possible models, which occur in 2 dimensions.

A general (Dirac) fermion in d dimensions has $2^{[d/2]}$ complex components, therefore in 2 dimensions it has 2 complex dimensions, and thus a Majorana fermion will have 2 real components. An on-shell Majorana fermion (that satisfies the Dirac equation, or equation of motion) will then have a single component.

Since we have a symmetry between bosons and fermions, the number of degrees of freedom of the bosons must match the number of degrees of freedom of the fermions (the symmetry will map a degree of freedom to another degree of freedom). This matching can be

- on-shell, in which case we have *on-shell supersymmetry* OR
- off-shell, in which case we have *off-shell supersymmetry*

Thus, in 2d, the simplest possible model has 1 Majorana fermion ψ (which has one degree of freedom on-shell), and 1 real scalar ϕ. We can then obtain **on-shell supersymmetry** and get the Wess-Zumino model in 2 dimensions.

The action of a free boson and a free fermion in two dimensions is

$$S = -\frac{1}{2}\int d^2x[(\partial_\mu\phi)^2 + \bar{\psi}\partial\!\!\!/\psi] \tag{3.8}$$

and this is actually the action of the free Wess-Zumino model. From the action, the mass dimension of the scalar is $[\phi] = 0$, and of the fermion is $[\psi] = 1/2$ (the mass dimension of $\int d^2x$ is -2 and of ∂_μ is $+1$, and the action is dimensionless).

To write down the supersymmetry transformation between the boson and the fermion, we start by varying the boson into fermion times ϵ, i.e

$$\delta\phi = \bar{\epsilon}\psi = \bar{\epsilon}_\alpha\psi^\alpha = \epsilon^\beta C_{\beta\alpha}\psi^\alpha \tag{3.9}$$

From this we infer that the mass dimension of ϵ is $[\epsilon] = -1/2$. By dimensional reasons, for the reverse transformation we must add an object of mass dimension 1 with no free vector indices, and the only one such object available to us is $\partial\!\!\!/$, thus

$$\delta\psi = \partial\!\!\!/\phi\epsilon \tag{3.10}$$

We can check that the above free action is indeed invariant on-shell under this symmetry. For this, we must use the Majorana spinor identities, valid both in 2d and 4d

$$\begin{aligned}&1)\ \bar{\epsilon}\chi = +\bar{\chi}\epsilon; \quad 2)\ \bar{\epsilon}\gamma_\mu\chi = -\bar{\chi}\gamma_\mu\epsilon\\ &3)\ \bar{\epsilon}\gamma_5\chi = +\bar{\chi}\gamma_5\epsilon \quad 4)\ \bar{\epsilon}\gamma_\mu\gamma_5\chi = +\bar{\chi}\gamma_\mu\gamma_5\epsilon\end{aligned} \tag{3.11}$$

To prove, for instance, the first identity, we write $\bar{\epsilon}\chi = \epsilon^\alpha C_{\alpha\beta}\chi^\beta$, but $C_{\alpha\beta}$ is antisymmetric and ϵ and χ anticommute, being spinors, thus we get $-\chi^\beta C_{\alpha\beta}\epsilon^\alpha = +\chi^\beta C_{\beta\alpha}\epsilon^\alpha$. Then the variation of the action gives

$$\delta S = -\int d^2x[-\phi\Box\delta\phi + \frac{1}{2}\delta\bar{\psi}\partial\!\!\!/\psi + \frac{1}{2}\bar{\psi}\partial\!\!\!/\delta\psi] = -\int d^2x[-\phi\Box\delta\phi + \bar{\psi}\partial\!\!\!/\delta\psi] \tag{3.12}$$

where in the second equality we have used partial integration together with identity 2) above. Then substituting the transformation law we get

$$\delta S = \int d^2x[-\phi\Box\bar{\epsilon}\psi + \bar{\psi}\partial\!\!\!/\partial\!\!\!/\epsilon] \tag{3.13}$$

But we have

$$\partial\!\!\!/\partial\!\!\!/ = \partial_\mu\partial_\nu\gamma^\mu\gamma^\nu = \partial_\mu\partial_\nu\frac{1}{2}\{\gamma_\mu,\gamma_\nu\} = \partial_\mu\partial_\nu g^{\mu\nu} = \Box \tag{3.14}$$

and by using this identity, together with two partial integrations, we obtain that $\delta S = 0$. So the action is invariant without the need for the equations of motion, so it would seem that this is an off-shell supersymmetry. However, the invariance of the action is not enough, since we have not proven that the above transformation law closes on the fields, i.e. that by acting twice on every field and forming the Lie algebra of the symmetry, we get back to the same field, or that we have a *representation of the Lie algebra* on the fields. The graded Lie algebra of supersymmetry is generically of the type

$$\{Q^i_\alpha, Q^j_\beta\} = 2(C\gamma^\mu)_{\alpha\beta}P_\mu\delta^{ij} + ... \tag{3.15}$$

In the case of a single supersymmetry, for the 2d Wess-Zumino model we don't have any +..., the above algebra is complete. In order to realize it on the fields, we need that (since P_μ is represented by the translation ∂_μ and Q_α is represented by δ_{ϵ_α})

$$[\delta_{\epsilon_{1,\alpha}}, \delta_{\epsilon_{2\beta}}]\begin{pmatrix}\phi\\ \psi\end{pmatrix} = 2\bar{\epsilon}_2\gamma^\mu\epsilon_1\partial_\mu\begin{pmatrix}\phi\\ \psi\end{pmatrix} \tag{3.16}$$

We get that

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\phi = 2\bar{\epsilon}_2\gamma^\rho\epsilon_1\partial_\rho\phi \tag{3.17}$$

as expected, but

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi = 2(\bar{\epsilon}_2\gamma^\rho\epsilon_1)\partial_\rho\psi - (\bar{\epsilon}_2\gamma^\rho\epsilon_1)\gamma_\rho\partial\!\!\!/\psi \tag{3.18}$$

thus we have an extra term that vanishes on-shell ($\partial\!\!\!/\psi = 0$). So on-shell the algebra is satisfied and we have on-shell supersymmetry.

It is left as an exercise to prove these relations. One must use the previous spinor identities together with new ones, called 2 dimensional "Fierz identities" (or "Fierz recoupling"),

$$M\chi(\bar{\psi}N\phi) = -\sum_j \frac{1}{2} MO_jN\phi(\bar{\psi}O_j\chi) \tag{3.19}$$

where M and N are arbitrary matrices, and the set of matrices $\{O_j\}$ is $= \{1, \gamma_\mu, \gamma_5\}$ (in 2 Minkowski dimensions, $\gamma_\mu = (i\tau_1, \tau_2)$ and $\gamma_5 = \tau_3$, where τ_i are Dirac matrices).

Off-shell supersymmetry

In 2 dimensions, an off-shell Majorana fermion has 2 degrees of freedom, but a scalar has only one. Thus to close the algebra of the Wess-Zumino model off-shell, we need one extra scalar field F. But on-shell, we must get back the previous model, thus the extra scalar F needs to be auxiliary (non-dynamical, with no propagating degree of freedom). That means that its action is $\int F^2/2$, thus

$$S = -\frac{1}{2}\int d^2x[(\partial_\mu\phi)^2 + \bar{\psi}\partial\!\!\!/\psi - F^2] \tag{3.20}$$

From the action we see that F has mass dimension $[F] = 1$, and the equation of motion of F is $F = 0$. The off-shell Wess-Zumino model algebra does not close on ψ, thus we need to add to $\delta\psi$ a term proportional to the equation of motion of F. By dimensional analysis, $F\epsilon$ has the right dimension. Since F itself is a (bosonic) equation of motion, its variation δF should be the fermionic equation of motion, and by dimensional analysis $\bar{\epsilon}\partial\!\!\!/\psi$ is OK. Thus the transformations laws are

$$\delta\phi = \bar{\epsilon}\psi; \quad \delta\psi = \partial\!\!\!/\phi\epsilon + F\epsilon; \quad \delta F = \bar{\epsilon}\partial\!\!\!/\psi \tag{3.21}$$

We can similarly check that these transformations leave the action invariant again, and moreover now we have

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\begin{pmatrix}\phi\\ \psi\\ F\end{pmatrix} = 2\bar{\epsilon}_2\gamma^\mu\epsilon_1\partial_\mu\begin{pmatrix}\phi\\ \psi\\ F\end{pmatrix} \tag{3.22}$$

so the algebra closes off-shell, i.e. we have an off-shell representation of $\{Q_\alpha, Q_\beta\} = 2(C\gamma^\mu)_{\alpha\beta}P_\mu$.

4 dimensions

Similarly, in 4 dimensions the on-shell Wess-Zumino model has one Majorana fermion, which however now has 2 real on-shell degrees of freedom, thus needs 2 real scalars, A and B. The action is then

$$S_0 = -\frac{1}{2}\int d^4x[(\partial_\mu A)^2 + (\partial_\mu B)^2 + \bar{\psi}\partial\!\!\!/\psi] \tag{3.23}$$

and the transformation laws are as in 2 dimensions, except now B aquires a γ_5 to distinguish it from A, thus

$$\delta A = \bar{\epsilon}\psi; \quad \delta B = \bar{\epsilon}i\gamma_5\psi; \quad \delta\psi = \partial\!\!\!/(A + i\gamma_5 B)\epsilon \tag{3.24}$$

And again, off-shell the Majorana fermion has 4 degrees of freedom, so one needs to introduce one auxiliary scalar for each propagating scalar, and the action is

$$S = S_0 + \int d^4x[\frac{F^2}{2} + \frac{G^2}{2}] \tag{3.25}$$

with the transformation rules

$$\begin{aligned}&\delta A = \bar{\epsilon}\psi; \quad \delta B = \bar{\epsilon}i\gamma_5\psi; \quad \delta\psi = \partial\!\!\!/(A + i\gamma_5 B)\epsilon + (F + i\gamma_5 G)\epsilon\\ &\delta F = \bar{\epsilon}\partial\!\!\!/\psi; \quad \delta G = \bar{\epsilon}i\gamma_5\partial\!\!\!/\psi\end{aligned} \tag{3.26}$$

One can form a complex field $\phi = A + iB$ and one complex auxiliary field $M = F + iG$, thus the Wess-Zumino multiplet in 4 dimensions is (ϕ, ψ, M).

We have written the free Wess-Zumino model in 2d and 4d, but one can write down interactions between them as well, that preserve the supersymmetry.

These were examples of $\mathcal{N} = 1$ supersymmetry that is, there was only one supersymmetry generator Q_α. The possible on-shell multiplets of $\mathcal{N} = 1$ supersymmetry that have spins ≤ 1 are

- The Wess-Zumino or chiral multiplet that we discussed, (ϕ, ψ).
- The vector multiplet (λ^A, A_μ^A), where A is an adjoint index. The vector A_μ in 4 dimensions has 2 on-shell degrees of freedom: it has 4 components, minus one gauge invariance symmetry parametrized by an arbitrary ϵ^a, $\delta A_\mu^A = \partial_\mu\epsilon^A$ giving 3 off-shell components. In the covariant gauge $\partial^\mu A_\mu = 0$ the equation of motion $k^2 = 0$ is supplemented with the constraint $k^\mu\epsilon_\mu^a(k) = 0$ ($\epsilon_\mu^a(k)$ =polarization), which has only 2 independent solutions. The two degrees of freedom of the gauge field match the 2 degrees of freedom of the on-shell fermion.

For $\mathcal{N} \geq 2$ supersymmetry, we have Q_α^i with $i = 1, ..., \mathcal{N}$. For $\mathcal{N} = 2$, the possible multiplets of spins ≤ 1 are

- The $\mathcal{N} = 2$ vector multiplet, made of one $\mathcal{N} = 1$ vector multiplet (A_μ, λ) and one $\mathcal{N} = 1$ chiral (Wess-Zumino) multiplet (ψ, ϕ).
- The $\mathcal{N} = 2$ hypermultiplet, made of two $\mathcal{N} = 1$ chiral multiplets (ψ_1, ϕ_1) and (ψ_2, ϕ_2).

For $\mathcal{N} = 4$ supersymmetry, there is a single multiplet of spins ≤ 1, the $\mathcal{N} = 4$ vector multiplet, made of an $\mathcal{N} = 2$ vector multiplet and a $\mathcal{N} = 2$ hypermultiplet, or one $\mathcal{N} = 1$ vector multiplet (A_μ, ψ_4) and 3 $\mathcal{N} = 1$ chiral multiplets $(\psi_i, \phi_1), i = 1, 2, 3$. They can be rearranged into $(A_\mu^a, \psi^{ai}, \phi_{[ij]})$, where $i = 1, .., 4$ is an $SU(4) = SO(6)$ index, $[ij]$ is the 6 dimensional antisymmetric representation of SU(4) or the fundamental representation of SO(6), and i is the fundamental representation of SU(4) or the spinor representation of SO(6). The field $\phi_{[ij]}$ has complex entries but satisfies a reality condition,

$$\phi_{ij}^\dagger = \phi^{ij} \equiv \epsilon^{ijkl}\phi_{kl} \tag{3.27}$$

The action of the $\mathcal{N}=1$ vector multiplet is

$$S_{\mathcal{N}=1SYM} = \int d^4x tr[-\frac{1}{4}F_{\mu\nu}^2 - \bar{\lambda}\not{D}\lambda(+\frac{D^2}{2})] \tag{3.28}$$

where $\not{D} = \gamma^\mu D_\mu$ and D is an auxiliary field for the off-shell action. It is just the action of a gauge field, a spinor minimally coupled to it, and an auxiliary field. The transformation rules are

$$\begin{aligned}
\delta A_\mu^a &= \bar{\epsilon}\gamma_\mu\psi^a \\
\delta\psi^a &= (-\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu}^a + i\gamma_5 D^a)\epsilon \\
\delta D^a &= i\bar{\epsilon}\gamma_5\not{D}\psi^a
\end{aligned} \tag{3.29}$$

They are similar to the rules of the Wess-Zumino multiplet, except for the gamma matrix factors introduced in order to match the index structure, and for replacing $\partial_\mu\phi$ with $F_{\mu\nu}$.

The action of the $\mathcal{N}=4$ Super Yang-Mills multiplet is

$$\begin{aligned}
S_{\mathcal{N}=4SYM} = &\int d^4x tr[-\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}\bar{\psi}_i\not{D}\psi^i - \frac{1}{2}D_\mu\phi_{ij}D^\mu\phi^{ij} \\
&-\frac{1}{2}\bar{\psi}^i[\phi_{ij},\psi^j] + \frac{1}{4}[\phi_{ij},\phi_{kl}][\phi^{ij},\phi^{kl}]]
\end{aligned} \tag{3.30}$$

where $D_\mu = \partial_\mu + g[A_\mu, \quad]$. This action however has no (covariant and un-constrained auxiliary fields) off-shell formulation.

The supersymmetry rules are

$$\begin{aligned}
\delta A_\mu^a &= \bar{\epsilon}_i\gamma_\mu\lambda^{ai} \\
\delta\phi_a^{ij} &= \bar{\epsilon}^{(i}\lambda_a^{j)} \\
\delta\lambda^{ai} &= -\frac{\gamma^{\mu\nu}}{2}F_{\mu\nu}^a\epsilon^i - \frac{1}{2}(\gamma^{mn})^i{}_j\partial_{(m}\phi^{a,kl}(\gamma_{n)})_{kl}\epsilon^j
\end{aligned} \tag{3.31}$$

Important concepts to remember

- A graded Lie algebra can contain the Poincare algebra, internal algebra and supersymmetry.
- The supersymmetry Q_α relates bosons and fermions.
- If the on-shell number of degrees of freedom of bosons and fermions match we have on-shell supersymmetry, if the off-shell number matches we have off-shell supersymmetry.
- For off-shell supersymmetry, the supersymmetry algebra must be realized on the fields.
- The prototype for all (linear) supersymmetry is the 2 dimensional Wess-Zumino model, with $\delta\phi = \bar{\epsilon}\psi, \delta\psi = \not{\partial}\phi\epsilon$.

- The Wess-Zumino model in 4 dimensions has a fermion and a complex scalar on-shell. Off-shell there is also an auxiliary complex scalar.

- The on-shell vector multiplet has a gauge field and a fermion

- The $\mathcal{N} = 4$ supersymmetric vector multiple ($\mathcal{N} = 4$ SYM) has one gauge field, 4 fermions and 6 scalars, all in the adjoint of the gauge field.

Exercises, section 3

1) Prove that the matrix

$$C_{AB} = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot\alpha\dot\beta} \end{pmatrix}; \epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.32}$$

is a representation of the 4d C matrix, i.e. $C^T = -C, C\gamma^\mu C^{-1} = -(\gamma^\mu)^T$, if γ^μ is represented by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}; \quad (\sigma^\mu)_{\alpha\dot\alpha} = (1, \vec\sigma)_{\alpha\dot\alpha}; \quad (\bar\sigma^\mu)^{\alpha\dot\alpha} = (1, -\vec\sigma)^{\alpha\dot\alpha} \tag{3.33}$$

2) Prove that if ϵ, χ are 4d Majorana spinors, we have

$$\bar\epsilon\gamma_\mu\gamma_5\chi = +\bar\chi\gamma_\mu\gamma_5\epsilon \tag{3.34}$$

3) Prove that, for

$$S = -\frac{1}{2}\int d^4x[(\partial_\mu\phi)^2 + \bar\psi\partial\!\!\!/\psi] \tag{3.35}$$

we have

$$\begin{aligned} [\delta_{\epsilon_1}, \delta_{\epsilon_2}]\phi &= 2\bar\epsilon_2\partial\!\!\!/\epsilon_1\phi \\ [\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi &= 2(\bar\epsilon_2\gamma^\rho\epsilon_1)\partial_\rho\psi - (\bar\epsilon_2\gamma^\rho\epsilon_1)\gamma_\rho\partial\!\!\!/\psi \end{aligned} \tag{3.36}$$

4) Show that the susy variation of the 4d Wess-Zumino model is zero, paralleling the 2d WZ model.

5) Check the invariance of the N=1 off-shell SYM action

$$S = \int d^4x[-\frac{1}{4}(F^a_{\mu\nu})^2 - \frac{1}{2}\bar\psi^a D\!\!\!\!/\psi_a + \frac{1}{2}D_a^2] \tag{3.37}$$

under the susy transformations

$$\delta A^a_\mu = \bar\epsilon\gamma_\mu\psi^a; \quad \delta\psi^a = (-\frac{1}{2}\sigma^{\mu\nu}F^a_{\mu\nu} + i\gamma_5 D^a)\epsilon; \quad \delta D^a = i\bar\epsilon\gamma_5 D\!\!\!\!/\psi^a \tag{3.38}$$

6)Calculate the number of off-shell degrees of freedom of the on-shell N=4 SYM action. Propose a set of bosonic+fermionic auxiliary fields that could make the number of degrees of freedom match. Are they likely to give an off-shell formulation, and why?

4 Basics of supergravity

Vielbeins and spin connections

We saw that gravity is defined by the metric $g_{\mu\nu}$, which in turn defines the Christoffel symbols $\Gamma^\mu{}_{\nu\rho}(g)$, which is like a gauge field of gravity, with the Riemann tensor $R^\mu{}_{\nu\rho\sigma}(\Gamma)$ playing the role of its field strength.

But there is a formulation that makes the gauge theory analogy more manifest, namely in terms of the "vielbein" e_μ^a and the "spin connection" ω_μ^{ab}. The word "vielbein" comes from the german viel= many and bein=leg. It was introduced in 4 dimensions, where it is known as "vierbein", since vier=four. In various dimensions one uses einbein, zweibein, dreibein,... (1,2,3= ein, zwei, drei), or generically vielbein, as we will do here.

Any curved space is locally flat, if we look at a scale much smaller than the scale of the curvature. That means that locally, we have the Lorentz invariance of special relativity. The vielbein is an object that makes that local Lorentz invariance manifest. It is a sort of square root of the metric, i.e.

$$g_{\mu\nu}(x) = e_\mu^a(x) e_\nu^b(x) \eta_{ab} \tag{4.1}$$

so in $e_\mu^a(x)$, μ is a "curved" index, acted upon by a general coordinate transformation (so that e_μ^a is a covariant vector of general coordinate transformations, like a gauge field), and a is a newly introduced "flat" index, acted upon by a local Lorentz gauge invariance. That is, around each point we define a small flat neighbourhood ("tangent space") and a is a tensor index living in that local Minkowski space, acted upon by Lorentz transformations.

We can check that an infinitesimal general coordinate transformation ("Einstein" transformation) $\delta x^\mu = \xi^\mu$ acting on the metric gives

$$\delta_\xi g_{\mu\nu}(x) = (\xi^\rho \partial_\rho) g_{\mu\nu} + (\partial_\mu \xi^\rho) g_{\rho\nu} + (\partial_\nu \xi^\rho) g_{\rho\nu} \tag{4.2}$$

where the first term corresponds to a translation, but there are extra terms. Thus the general coordinate transformations are the general relativity analog of P_μ translations in special relativity.

On the vielbein e_μ^a, the infinitesimal coordinate transformation gives

$$\delta_\xi e_\mu^a(x) = (\xi^\rho \partial_\rho) e_\mu^a + (\partial_\mu \xi^\rho) e_\rho^a \tag{4.3}$$

thus it acts only on the curved index μ. On the other hand, the local Lorentz transformation

$$\delta_{l.L.} e_\mu^a(x) = \lambda^a{}_b(x) e_\mu^b(x) \tag{4.4}$$

is as usual.

Thus the vielbein is like a sort of gauge field, with one covariant vector index and a gauge group index. But there is one more "gauge field" ω_μ^{ab}, the "spin connection", which is defined as the "connection" ($\equiv$ gauge field) for the action of the Lorentz group on spinors.

Namely, the curved space covariant derivative acting on spinors acts similarly to the gauge field covariant derivative on a spinor, by

$$D_\mu \psi = \partial_\mu \psi + \frac{1}{4} \omega_\mu^{ab} \Gamma^{ab} \psi \tag{4.5}$$

This definition means that $D_\mu\psi$ is the object that transforms as a tensor under general coordinate transformations. It implies that ω_μ^{ab} acts as a gauge field on any local Lorentz index.

If there are no *dynamical* fermions (i.e. fermions that have a kinetic term in the action) then $\omega_\mu^{ab}=\omega_\mu^{ab}(e)$ is a fixed function, defined through the "vielbein postulate"

$$T^a_{[\mu\nu]}=D_{[\mu}e^a_{\nu]}=\partial_{[\mu}e^a_{\nu]}+\omega^{ab}_{[\mu}e^b_{\nu]}=0 \tag{4.6}$$

Note that we can also start with

$$D_\mu e_\nu^a\equiv\partial_\mu e_\nu^a+\omega_\mu^{ab}e_\nu^b-\Gamma^\rho{}_{\mu\nu}e^a_\rho=0 \tag{4.7}$$

ant antisymmetrize, since $\Gamma^\rho{}_{\mu\nu}$ is symmetric. This is also sometimes called the vielbein postulate.

Here T^a is called the "torsion", and as we can see it is a sort of field strength of e^a_μ, and the vielbein postulate says that the torsion (field strength of vielbein) is zero.

But we can also construct an object that is a field strength of ω_μ^{ab},

$$R^{ab}_{\mu\nu}(\omega)=\partial_\mu\omega_\nu^{ab}-\partial_\nu\omega_\mu^{ab}+\omega_\mu^{ab}\omega_\nu^{bc}-\omega_\nu^{ac}\omega_\mu^{cb} \tag{4.8}$$

and this time the definition is exactly the definition of the field strength of a gauge field of the Lorentz group (though there still are subtleties in trying to make the identification of ω_μ^{ab} with a gauge field of the Lorentz group).

This curvature is in fact the analog of the Riemann tensor, i.e. we have

$$R^{ab}_{\rho\sigma}(\omega(e))=e^a_\mu e^{-1,\nu b}R^\mu{}_{\nu\rho\sigma}(\Gamma(e)) \tag{4.9}$$

The Einstein-Hilbert action is then

$$S_{EH}=-\frac{1}{16\pi G}\int d^4x(\det e)R^{ab}_{\mu\nu}(\omega(e))e_a^{-1,\mu}e^{-1,\nu}|_b \tag{4.10}$$

since $\sqrt{detg}=\det e$.

The formulation just described of gravity in terms of e and ω is the *second order formulation*, so called because ω is not independent, but is a function of e.

But notice that if we make ω an independent variable in the above Einstein-Hilbert action, the ω equation of motion gives exactly $T^a_{\mu\nu}=0$, i.e. the vielbein postulate that we needed to postulate before. Thus we might as well make ω independent without changing the classical theory (only possibly the quantum version). This is then the *first order formulation* of gravity (Palatini formalism), in terms of $(e^a_\mu,\omega_\mu^{ab})$.

Supergravity

Supergravity can be defined in two independent ways that give the same result. It is a supersymmetric theory of gravity; and it is also a theory of local supersymmetry. Thus we could either take Einstein gravity and supersymmetrize it, or we can take a supersymmetric model and make the supersymmetry local. In practice we use a combination of the two.

We want a theory of local supersymmetry, which means that we need to make the rigid ϵ^α transformation local. We know from gauge theory that if we want to make a global symmetry local we need to introduce a gauge field for the symmetry. The gauge field would be "A_μ^α" (since the supersymmetry acts on the index α), which we denote in fact by $\psi_{\mu\alpha}$ and call the gravitino.

Here μ is a curved space index ("curved") and α is a local Lorentz spinor index ("flat"). In flat space, $\psi_{\mu\alpha}$ would have the same kind of indices and we can then show that it forms a spin 3/2 field, therefore the same is true in curved space.

The fact that we have a supersymmetric theory of gravity means that gravitino must be transformed by supersymmetry into some gravity variable, thus $\psi_\alpha = Q_\alpha(gravity)$. But the index structure tells us that the gravity variable cannot be the metric, but something with only one curved index, namely the vielbein.

Therefore we see that supergravity needs the vielbein-spin connection formulation of gravity. To write down the supersymmetry transformations, we start with the vielbein. In analogy with the Wess-Zumino model where $\delta\phi = \bar{\epsilon}\phi$ or the vector multiplet where the gauge field variation is $\delta A_\mu^a = \bar{\epsilon}\gamma_\mu\psi^a$, it is easy to see that the vielbein variation has to be

$$\delta e_\mu^a = \frac{k}{2}\bar{\epsilon}\gamma^a\psi_\mu \tag{4.11}$$

where k is the Newton constant. Since ψ is like a gauge field of local supersymmetry, we expect something like $\delta A_\mu = D_\mu\epsilon$. Therefore we must have

$$\delta\psi_\mu = \frac{1}{k}D_\mu\epsilon; \quad D_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab}\epsilon \tag{4.12}$$

plus maybe more terms.

The action for a free spin 3/2 field in flat space is the Rarita-Schwinger action which is

$$S_{RS} = -\frac{1}{2}\int d^4x\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu\partial_\rho\psi_\sigma = -\frac{1}{2}\int d^dx\bar{\psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\psi_\rho \tag{4.13}$$

where the first form is only valid in 4 dimensions and the second is valid in all dimensions ($\epsilon^{\mu\nu\rho\sigma}\gamma_4\gamma_\nu = \gamma^{\mu\rho\sigma}$ in 4 dimensions). In curved space, this becomes

$$S_{RS} = -\frac{1}{2}\int d^4x\epsilon^{\mu\nu\rho\sigma}\bar{\psi}_\mu\gamma_5\gamma_\nu D_\rho\psi_\sigma = -\frac{1}{2}\int d^dx(dete)\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho \tag{4.14}$$

$\mathcal{N}=1$ (on-shell) supergravity in 4 dimensions

We are now ready to write down $\mathcal{N}=1$ on-shell supergravity in 4 dimensions. Its action is just the sum of the Einstein-Hilbert action and the Rarita-Schwinger action

$$S_{\mathcal{N}=1} = S_{EH}(\omega, e) + S_{RS}(\psi_\mu) \tag{4.15}$$

and the supersymmetry transformations rules are just the ones defined previously,

$$\delta e_\mu^a = \frac{k}{2}\bar{\epsilon}\gamma^a\psi_\mu; \quad \delta\psi_\mu = \frac{1}{k}D_\mu\epsilon \tag{4.16}$$

However, this is not yet enough to specify the theory. We must specify the formalism and various quantities:

- second order formalism: The independent fields are $e_\mu 6a, \psi_\mu$. ω is not an independent field. But now there is a dynamical fermion (ψ_μ), so the torsion $T^a_{\mu\nu}$ is not zero anymore, thus $\omega \neq \omega(e)$! In fact,

$$\omega_\mu^{ab} = \omega_\mu^{ab}(e,\psi) = \omega_\mu^{ab}(e) + \psi\psi \ \text{ terms} \tag{4.17}$$

  is found by varying the action with respect to ω, as in the $\psi = 0$ case:

$$\frac{\delta S_{\mathcal{N}=1}}{\delta \omega_\mu^{ab}} = 0 \Rightarrow \omega_\mu^{ab}(e,\psi) \tag{4.18}$$

- first order formalism: All fields, ψ, e, ω are independent. But now we must supplement the action with a transformation law for ω. It is

$$\begin{aligned}\delta\omega_\mu^{ab}(\text{first order}) &= -\frac{1}{4}\bar{\epsilon}\gamma_5\gamma_\mu\tilde{\psi}^{ab} + \frac{1}{8}\bar{\epsilon}\gamma_5(\gamma^\lambda\tilde{\psi}^b_\lambda e^a_\mu - \gamma^\lambda\tilde{\psi}^a_\lambda e^b_\mu)\\ \tilde{\psi}^{ab} &= \epsilon^{abcd}\psi_{cd}; \quad \psi_{ab} = e_a^{-1\mu}e_b^{-1\nu}(D_\mu\psi_\nu - D_\nu\psi_\mu)\end{aligned} \tag{4.19}$$

General features of supergravity theories
4 dimensions

The $\mathcal{N}=1$ supergravity multiplet is $(e^a_\mu, \psi_{\mu\alpha})$ as we saw, and has spins (2,3/2).

It can also couple with other $\mathcal{N}=1$ supersymmetric multiplets of lower spin: the chiral multiplet of spins (1/2,0) and the gauge multiplet of spins (1,1/2) that have been described, as well as the so called gravitino multiplet, composed of a gravitino and a vector, thus spins (3/2,1).

By adding appropriate numbers of such multiplets we obtain the $\mathcal{N}=2,3,4,8$ supergravity multiplets. Here $\mathcal{N}$ is the number of supersymmetries, and since it acts on the graviton, there should be exactly $\mathcal{N}$ gravitini in the multiplet, so that each supersymetry maps the graviton to a different gravitino.

$\mathcal{N}=8$ supergravity is the maximal supersymmetric multiplet that has spins ≤ 2 (i.e., an $\mathcal{N}>8$ multiplet will contains spins >2, which are not very well defined), so we consider only $\mathcal{N}\leq 8$.

Coupling to supergravity of a supersymmetric multiplet is a generalization of coupling to gravity, which means putting fields in curved space. Now we put fields in curved space and introduce also a few more couplings.

We will denote the $\mathcal{N}=1$ supersymmetry multiplets by brackets, e.g. (1,1/2), (1/2,0), etc. The supergravity multiplets are compose of the following fields:

$\mathcal{N}=3$ supergravity: Supergravity multiplet (2,3/2) + 2 gravitino multiplets (3/2,1) + one vector multiplet (1,1/2). The fields are then $\{e^a_\mu, \psi^i_\mu, A^i_\mu, \lambda\}$, i=1,2,3.

$\mathcal{N}=4$ supergravity: Supergravity multiplet (2,3/2) + 3 gravitino multiplets (3/2,1) + 3 vector multiplets (1,1/2) + one chiral multiplet (1/2,0). The fields are $\{e^a_\mu, \psi^i_\mu, A^k_\mu, B^k_\mu, \lambda^i, \phi, B\}$, where i=1,2,3,4; k=1,2,3, A is a vector, B is an axial vector, ϕ is a scalar and B is a pseudoscalar.

$\mathcal{N}=8$ supergravity: Supergravity multiplet (2,3/2) + 7 gravitino multiplets (3/2,1) + 21 vector multiplets (1,1/2) + 35 chiral multiplets (1/2,0). The fields are $\{e^a_\mu, \psi^i_\mu, A^{IJ}_\mu, \chi_{ijk}, \nu\}$

which are: one graviton, 8 gravitinos ψ_μ^i, 28 photons A_μ^{IJ}, 56 spin 1/2 fermions χ_{ijk} and 70 scalars in the matrix ν.

In these models, the photons are not coupled to the fermions, i.e. the gauge coupling $g = 0$, thus they are "ungauged" models. But these models have *global* symmetries, e.g. the $\mathcal{N} = 8$ model has SO(8) global symmetry.

One can couple the gauge fields to the fermions, thus "gauging" (making local) some global symmetry (e.g. SO(8)). Thus abelian fields become nonabelian (Yang-Mills), i.e. self-coupled. Another way to obtain the gauged models is by adding a cosmological constant and requiring invariance

$$\delta\psi_\mu^i = D_\mu(\omega(e,\psi))\epsilon^i + g\gamma_\mu\epsilon^i + gA_\mu\epsilon^i \tag{4.20}$$

where g is related to the cosmological constant, i.e. $\Lambda \propto g$. Because of the cosmological constant, it means that gauged supergravities have Anti de Sitter (AdS) backgrounds.

Higher dimensions

In $D > 4$, it is possible to have also antisymmetric tensor fields $A_{\mu_1,\ldots,\mu_n}$, which are just an extension of abelian vector fields, with field strength

$$F_{\mu_1,\ldots,\mu_{n+1}} = \partial_{[\mu_1}A_{\mu_2,\ldots,\mu_{n+1}]} \tag{4.21}$$

and gauge invariance

$$\delta A_{\mu_1,\ldots,\mu_n} = \partial_{[\mu_1}\Lambda_{\mu_2,\ldots,\mu_n]} \tag{4.22}$$

and action

$$\int d^dx(\det e)F^2_{\mu_1,\ldots,\mu_{n+1}} \tag{4.23}$$

The maximal model possible that makes sense as a 4 dimensional theory is the $\mathcal{N} = 1$ supergravity model in 11 dimensions, made up of a graviton e_μ^a, a gravitino $\psi_{\mu\alpha}$ and a 3 index antisymmetric tensor $A_{\mu\nu\rho}$.

But how do we make sense of a higher dimensional theory? The answer is the so called **Kaluza-Klein (KK) dimensional reduction**. The idea is that the extra dimensions $(d-4)$ are curled up in a small space, like a small sphere or a small $d-4$-torus.

For this to happen, we consider a background solution of the theory that looks like, e.g. (in the simplest case) as a product space,

$$g_{\Lambda\Sigma} = \begin{pmatrix} g^{(0)}_{\mu\nu}(x) & 0 \\ 0 & g^{(0)}_{mn}(y) \end{pmatrix} \tag{4.24}$$

where $g^{(0)}_{\mu\nu}(x)$ is the metric on our 4 dimensional space and $g^{(0)}_{mn}(y)$ is the metric on the extra dimensional space.

We then expand the fields of the higher dimensional theory around this background solution in Fourier-like modes, called spherical harmonics. E.g., $g_{\mu\nu}(x,y) = g^{(0)}_{\mu\nu}(x) + \sum_n g^{(n)}_{\mu\nu}(x)Y_n(y)$, with $Y_n(y)$ being the spherical harmonic (like e^{ikx} for Fourier modes).

Finally, dimensional reduction means dropping the higher modes, and keeping only the lowest Fourier mode, the constant one, e.g.

$$g_{\Lambda\Sigma} = \begin{pmatrix} g^{(0)}_{\mu\nu}(x) + h_{\mu\nu}(x) & h_{\mu m}(x) \\ h_{m\nu}(x) & g^{(0)}_{mn}(y) + h_{mn}(x) \end{pmatrix} \tag{4.25}$$

Important concepts to remember

- Vielbeins are defined by $g_{\mu\nu}(x) = e^a_\mu(x) e^b_\nu(x) \eta_{ab}$, by introducing a Minkowski space in the neighbourhood of a point x, giving local Lorentz invariance.
- The spin connection is the gauge field needed to define covariant derivatives acting on spinors. In the absence of dynamical fermions, it is determined as $\omega = \omega(e)$ by the vielbein postulate: the torsion is zero.
- The field strength of this gauge field is related to the Riemann tensor.
- In the first order formulation (Palatini), the spin connection is independent, and is determined from its equation of motion.
- Supergravity is a supersymmetric theory of gravity and a theory of local supersymmetry.
- The gauge field of local supersymmetry and superpartner of the vielbein (graviton) is the gravitino ψ_μ.
- Supergravity (local supersymmetry) is of the type $\delta e^a_\mu = (k/2)\bar{\epsilon}\gamma^a\psi_\mu + ...$, $\delta\psi_\mu = (D_\mu\epsilon)/k + ...$
- For each supersymmetry we have a gravitino. The maximal supersymmetry in d=4 is $\mathcal{N} = 8$.
- Supergravity theories in higher dimensions can contain antisymmetric tensor fields.
- The maximal dimension for a supergravity theory is d=11, with a unique model composed of $e^a_\mu, \psi_\mu, A_{\mu\nu\rho}$.
- A higher dimensional theory can be dimensionally reduced: expand in generalized Fourier modes (spherical harmonics) around a vacuum solution that contains a compact space for the extra dimensions (like a sphere or torus), and keep only the lowest modes.

Exercises, section 4

1) Prove that the general coordinate transformation on $g_{\mu\nu}$,

$$g'_{\mu\nu}(x') = g_{\rho\sigma}(x)\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} \tag{4.26}$$

reduces for infinitesimal tranformations to

$$\partial_\xi g_{\mu\nu}(x) = (\xi^\rho\partial_\rho)g_{\mu\nu} + (\partial_\mu\xi^\rho)g_{\rho\sigma} + (\partial_\nu\xi^\rho)g_{\rho\mu} \tag{4.27}$$

2) Check that

$$\omega_\mu^{ab}(e) = \frac{1}{2}e^{a\nu}(\partial_\mu e_\nu^b - \partial_\nu e_\mu^b) - \frac{1}{2}e^{b\nu}(\partial_\mu e_\nu^a - \partial_\nu e_\mu^a) - \frac{1}{2}e^{a\rho}e^{b\sigma}(\partial_\rho e_{c\sigma} - \partial_\sigma e_{c\rho})e_\mu^c \tag{4.28}$$

satisfies the no-torsion (vielbein) constraint, $T^a_{\mu\nu} = D_{[\mu}e^a_{\nu]} = 0$.

3) Check that the equation of motion for ω_μ^{ab} in the first order formulation of gravity (Palatini formalism) gives $T^a_{\mu\nu} = 0$.

4) Write down the free gravitino equation of motion in curved space.

5) Find $\omega_\mu^{ab}(e,\psi) - \omega_\mu^{ab}(e)$ in the second order formalism for N=1 supergravity.

6) Calculate the number of off-shell bosonic and fermionic degrees of freedom of N=8 on-shell supergravity.

7) Consider the Kaluza Klein dimensional reduction ansatz from 5d to 4d

$$g_{\Lambda\Pi} = \phi^{-1/3}\begin{pmatrix} \eta_{\mu\nu} + h_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{pmatrix} \tag{4.29}$$

Show that the action for the linearized perturbation $h_{\mu\nu}$ contains no factors of ϕ. (Hint: first show that for small $h_{\mu\nu}$, where $g_{\mu\nu} = f(\eta_{\mu\nu} + h_{\mu\nu})$, $R_{\mu\nu}(g)$ is independent of f).